Math 311-102

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Example: image and null space

If $f(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 6 & 0 \\ 0 & -8 & 16 \end{pmatrix}$, find the null space of f and find the image of f.

Null space. Solve $\begin{pmatrix} 1 & 0 & 3 \\ 4 & 6 & 0 \\ 0 & -8 & 16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$. Row reduce to get $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$. Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ where t is an arbitrary number. The null space is a line.

Another interpretation: The null space consists of all vectors that are orthogonal to every row of the matrix A.

Example continued

The *image* consists of all linear combinations of the *columns* of the matrix A. We can find the image by *column* reducing:

$$\begin{pmatrix} 1 & 0 & 3 \\ 4 & 6 & 0 \\ 0 & -8 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 4 & 6 & -12 \\ 0 & -8 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 0 & -4 & 0 \end{pmatrix}.$$

The image can be described as all vectors $t(\vec{i} + 4\vec{j}) + s(3\vec{j} - 4\vec{k})$ where t and s are arbitrary numbers.

Another interpretation: The image is the plane with equation $-16x + 4y + 3z = 0$.

Shortcut. *The dimension of the image and the dimension of the null space add up to the dimension of the domain.* Having shown that the dimension of the null space is 1, we can deduce that the dimension of the image is 2, so the image consists of all linear combinations of any two independent columns (for instance, the first two columns).

Another example

Suppose $f(\vec{x}) = A\vec{x}$, and the reduced row echelon form of A is $\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 6 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Say as much as possible about the domain, range, null space, and image of f.

Solution. The domain is $\mathbb{R}^7$ and the range is $\mathbb{R}^5$. For the null space, we can choose x_2, x_4, x_6, and x_7 arbitrarily, and then x_1, x_3, and x_5 (the leading variables) will be determined. The null space is 4-dimensional and consists of the vectors $x_2(-2, 1, 0, 0, 0, 0, 0) + x_4(-3, 0, -1, 1, 0, 0, 0) + x_6(-6, 0, -5, 0, -4, 1, 0) + x_7(0, 0, -7, 0, 0, 0, 1)$.
The image is 3-dimensional: all linear combinations of columns 1, 3, and 5 of the original matrix (not the row-reduced matrix).

Basis

A *basis* for a vector space is a set of vectors satisfying any one of the following equivalent descriptions:
1. a maximal linearly independent set;
2. a minimal spanning set;
3. a set such that every vector can be represented in a unique way as a linear combination of elements of the basis.

Examples. The standard basis for $\mathbb{R}^2$ is the pair of vectors $(1, 0)$ and $(0, 1)$.
Another basis for $\mathbb{R}^2$ is the pair of vectors $(1, 1)$ and $(1, -1)$.
The standard basis for the space $\mathcal{P}_2$ of polynomials of degree less than or equal to 2 is 1, x, x^2.
Another basis for $\mathcal{P}_2$ is the set of so-called Legendre polynomials 1, x, and $\frac{1}{2}(3x^2 - 1)$.

The *dimension* of a vector space is the number of elements in a basis. All bases have the same number of elements.

Example: bases

The set of all linear combinations of $\sin(x)$ and $\cos(x)$ is a two-dimensional vector space (a subspace of the vector space of differentiable functions). For the basis $\{\sin(x), \cos(x)\}$, what matrix represents the linear operator of differentiation?

Solution. Since the derivative of the first basis element equals the second, and the derivative of the second basis element equals the negative of the first, the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Continuation. Euler's formula says $e^{ix} = \cos(x) + i\sin(x)$ (where i is the complex number whose square is -1). What matrix represents the operator of differentiation with respect to the alternate basis $\{e^{ix}, e^{-ix}\}$? Answer: $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

A linear transformation is easiest to understand if a basis is used in which the representing matrix is diagonal.