Math 311-102

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Inner products

An *inner product* on a vector space is a generalization of the dot product on $\mathbb{R}^n$: a function that takes pairs of vectors as inputs and produces a real number as output. The notation for the inner product of vectors $\vec{v}$ and $\vec{w}$ is $\langle \vec{v}, \vec{w} \rangle$ or $\langle \vec{v} \mid \vec{w} \rangle$.

An inner product is required to have the following properties.
Symmetry: $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$.
Linearity: $\langle (\vec{u} + \vec{v}), \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ and $\langle (a\vec{u}), \vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle$.
Positivity: $\langle \vec{v}, \vec{v} \rangle > 0$ if $\vec{v} \neq \vec{0}$.

Examples. 1. The usual dot product on $\mathbb{R}^3$:
$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2 + x_3y_3$.
2. The weighted product on $\mathbb{R}^3$:
$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + 5x_2y_2 + 7x_3y_3$.
3. Integration on the space $\mathcal{P}$ of polynomials:
$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

Norms

Every inner product has an associated *norm* $\|\vec{v}\|$ (a generalization of length) defined by $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

Example. For the inner product of integration,
$\|p(x)\| = \left(\int_{-1}^{1} p(x)^2\,dx\right)^{1/2}$.

Similarly, every inner product has an associated notion of angle between vectors: $\cos(\theta) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\|\,\|\vec{w}\|}$.

Example. The "angle" between the functions x^2 and x^3 (for the inner product of integration) is determined by
$\cos(\theta) = \frac{\int_{-1}^{1} x^5\,dx}{(\int_{-1}^{1} x^4\,dx)^{1/2}(\int_{-1}^{1} x^6\,dx)^{1/2}} = 0$, so these two functions are orthogonal.

Every inner product satisfies the *Cauchy-Schwarz inequality* $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\,\|\vec{w}\|$ (corresponding to $|\cos(\theta)| \leq 1$).

Orthonormal bases

The nicest kind of basis consists of orthogonal vectors with norm (length) equal to 1.

Examples. 1. The standard basis $\{(1,0), (0,1)\}$ in $\mathbb{R}^2$ is orthonormal (for the standard inner product).

2. The basis $\{(1,1), (1,-1)\}$ is orthogonal, but not normalized.
The basis $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$ is orthonormal.

3. In the space $\mathcal{P}_2$ of polynomials, the basis $\{1, x, x^2\}$ is not orthogonal because $\langle 1, x^2 \rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \neq 0$.
The basis $\{1, x, (x^2 - \frac{1}{3})\}$ is orthogonal but not orthonormal.
The basis $\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{5}{8}}(3x^2 - 1)\}$ is orthonormal.

Gram-Schmidt orthonormalization

There is a standard procedure for producing an orthonormal basis related to an arbitrary basis.

Example. Starting from the basis vectors $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (0,1,1)$, and $\vec{v}_3 = (1,1,1)$, produce an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Recursive procedure. Subtract from each vector its projection on the previously constructed vectors, and normalize.
First step: Normalize $\vec{v}_1$ to get $\vec{u}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$.
Second step: Subtract from $\vec{v}_2$ its projection on $\vec{u}_1$ to get $\vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 = (-\frac{1}{2}, \frac{1}{2}, 1)$; normalize to get $\vec{u}_2 = (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}})$.
Third step: Subtract from $\vec{v}_3$ its projections on $\vec{u}_1$ and $\vec{u}_2$ to get $\vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 = (1,1,1) - (1,1,0) - (-\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$
$= (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$; normalize to get $\vec{u}_3 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Example continued

The vectors $\vec{u}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\vec{u}_2 = (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}})$, and $\vec{u}_3 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ form an orthonormal basis for $\mathbb{R}^3$. Write the vector $(4,-2,3)$ as linear combination of these basis vectors: namely, $(4,-2,3) = a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3$.

Solution. Take the inner product of both sides with $\vec{u}_1$ to get $\langle (4,-2,3), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \rangle = a_1$ or $a_1 = \sqrt{2}$.
Similarly, $a_2 = \langle (4,-2,3), \vec{u}_2 \rangle = 0$, and $a_3 = \langle (4,-2,3), \vec{u}_3 \rangle$ $= 3\sqrt{3}$. So $\vec{v} = \sqrt{2}\,\vec{u}_1 + 3\sqrt{3}\,\vec{u}_3$.

General principle for orthonormal systems. If $\vec{v} = \sum_k a_k \vec{u}_k$, and if the vectors $\vec{u}_k$ are orthonormal, then the coefficients a_k can be read off via $a_k = \langle \vec{v}, \vec{u}_k \rangle$.