Math 311-102

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About the exam

The second examination is tomorrow, Thursday, June 23.

Please bring paper (or a bluebook) to the exam.

The exam covers everything on the syllabus to date since the first exam.

There are 10 questions on the exam.

What are the main topics?

Image and null space; basis and dimension

Sample problems: (a) If $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 5 & 6 & 9 \end{pmatrix}$, find a basis for the image and a basis for the null space.

(b) If $B = \begin{pmatrix} 1 & 2 & t \\ 2 & 4 & 5 \\ 3 & 6 & 10 \end{pmatrix}$, for which value(s) of t does the null space of B have dimension 0? 1? 2? 3?

(c) Give an example of a linear transformation $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the null space has dimension 1 and the vector $(1, 0, 1)$ is in the image.

Eigenvalues and eigenvectors

Sample problems: (a) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$.

(b) For which value(s) of t, if any, does the matrix $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & t \end{pmatrix}$ have three linearly independent eigenvectors?

(c) Express the vector $(1, 0, 0)$ as a linear combination of the eigenvectors of the matrix $\begin{pmatrix} -3 & 30 & -60 \\ 2 & -20 & 40 \\ -1 & -14 & 22 \end{pmatrix}$.

Orthonormal bases

Sample problems: (a) Find an orthonormal basis for $\mathbb{R}^3$ in which one of the basis vectors is $(\frac{2}{7}, \frac{3}{7}, \frac{6}{7})$.

(b) Starting from the functions 1, x, and x^2, use the Gram-Schmidt procedure to construct an orthonormal set with respect to the inner product $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$.

(c) Find an orthonormal basis for $\mathbb{R}^3$ in which two of the basis vectors span the same plane as do the vectors $(1,1,0)$ and $(1,0,1)$.

Space curves

Sample problems: (a) If $f(t) = (te^t, t^2\cos(t), e^t\sin^2(t))$, find an equation for the line tangent to the curve at the point where $t = 0$.

(b) If $f(t) = (t, t^2, t^3)$, is there a point on the curve such that the tangent line at that point passes through the origin?

(c) If $f(t) = (2t, t^2 + 1, t^3)$, either find two points on the curve whose tangent vectors are orthogonal or show that no such points exist.

Surfaces

Sample problems: (a) If $g(u,v) = (u^2\cos(v), uv^2, ue^v)$, find an equation for the tangent plane to the surface at the point where $u = 1$ and $v = 0$.

(b) If $g(u,v) = (u, v, uv^2)$, find an orthonormal basis for $\mathbb{R}^3$ such that two of the basis vectors are tangent to the surface at the point where $u = 1$ and $v = 2$.

(c) Find a 3×3 matrix A such that two of its eigenvectors are tangent to the surface defined by $g(u,v) = (v\sin(u), 2u + e^v, u + 3v)$ at the point on the surface where $u = 0$ and $v = 0$.

Directional derivative

Sample problems: (a) If $f(x,y,z) = x^2 + xy + yz^3$, find the directional derivative of f in the direction of the unit vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$ at the point $(1,2,3)$.

(b) Find the directional derivative of $f(x,y,z) = xe^{yz} + y^2$ in the direction of an eigenvector of the matrix $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix}$.

(c) If $f(x,y,z) = x\cos(y) + y\cos(x) + xyz$, in what direction is the directional derivative maximal at the point $(0,0,0)$?

The derivative matrix

Sample problems: (a) If $f(u,v) = \begin{pmatrix} ue^v \\ u+v \\ \cos(v) \end{pmatrix}$, find the derivative matrix of f.

(b) If $f(x,y,z) = \begin{pmatrix} x^2+xy+z \\ xy \\ 3x+4y+5z \end{pmatrix}$, find the eigenvalues of the derivative matrix at $(0,0,0)$.

(c) Give an example of a transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the transformation is locally invertible near the point $(0,0,0)$ but is not locally invertible near the point $(1,1,1)$.

Chain rule

Sample problems: (a) If $f(x,y,z) = x^2+y^2+z^2$, and (r,θ,z) represent cylindrical coordinates, find $\dfrac{\partial f}{\partial \theta}$.

(b) If $f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^2v^2 \\ 2u+3v \end{pmatrix}$ and $\begin{pmatrix} u \\ v \end{pmatrix} = g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xe^y \\ ye^x \end{pmatrix}$, find the derivative matrix of the composite function $f \circ g$.

(c) Suppose $\begin{pmatrix} u \\ v \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix}$ is an invertible coordinate transformation in $\mathbb{R}^2$. True or false: $\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u} = 1$.

Change of variables in integrals

Sample problems: (a) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx$.

(b) Evaluate $\int_B z^2\,dx\,dy\,dz$, where B is the ball defined by $x^2+y^2+z^2 \leq 4$.

(c) Use the coordinate transformation $u = e^x\cos(y)$, $v = e^x\sin(y)$ to evaluate the integral $\int_R \sqrt{u^2+v^2}\,du\,dv$, where R is the region in the uv-plane corresponding to the region in the xy-plane defined by $0 \leq x \leq 1$ and $0 \leq y \leq \pi/2$.