Math 311-102

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Line (path) integrals

Example. A particle moves under the influence of a force $\vec{F} = 6x^2y\vec{\imath} + 10xy^2\vec{\jmath}$ along a path C described in parametric form by $g(t) = (t, t^3)$ as t goes from 0 to 1. Find the work done by the force.

Solution. Compute $\int_C \vec{F} \cdot d\vec{x}$, where $d\vec{x}$ is the vector (dx, dy).
$\int_C \vec{F} \cdot d\vec{x} = \int_C 6x^2y\,dx + 10xy^2\,dy = \int_0^1 6t^5\,dt + 10t^7 \times 3t^2\,dt$
$= 1 + 3 = 4.$

Example. Compute $\int_C (x+y)\,dx + (x-y)\,dy$, where C is the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$ (oriented counterclockwise).

Solution. Choose parametrizations for each leg of the triangle. For the hypotenuse, you could use the parametrization $y = t$, $x = 1 - t$. The integral becomes the sum
$\int_0^1 x\,dx + \int_0^1 1 \times (-dt) + (1-2t)\,dt + \int_1^0 -y\,dy = \frac{1}{2} - 1 + \frac{1}{2} = 0.$

Green's theorem (in the plane)

$$\int_{\substack{\text{closed}\\\text{curve}}} P(x,y)\,dx + Q(x,y)\,dy = \iint_{\substack{\text{region}\\\text{inside}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy$$

where the curve is traversed in the direction that leaves the region on the left-hand side.

Example. For the preceding example with the triangle,
$\int_C (x+y)\,dx + (x-y)\,dy = \iint_{\text{inside}} \left(\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial y}(x+y)\right)\,dx\,dy$
$= \iint_{\text{inside}} (1-1)\,dx\,dy = 0.$

Example. Let C be the circle centered at $(0,0)$ with radius 2, oriented counterclockwise. Compute $\int_C y\,dx - x\,dy$.

Solution. You *could* parametrize the path via $x = 2\cos(\theta)$, $y = 2\sin(\theta)$ and evaluate the line integral. Using Green's theorem instead gives $\iint_{\substack{\text{inside}\\\text{of circle}}} (-1-1)\,dx\,dy$
$= -2 \times (\text{area of circle}) = -8\pi.$

Stokes's theorem (in the plane)

Reinterpretation of $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$. Write $\nabla = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y}$, a vector differential operator. Write $\vec{F} = \vec{\imath}P + \vec{\jmath}Q$, a vector force.
Then $(\nabla \times \vec{F}) \cdot \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$.

Reinterpretation of $\vec{F} \cdot d\vec{x}$. If our path is described in parametric form as $g(t) = (g_1(t), g_2(t))$, then $d\vec{x} = \vec{\imath}\,dx + \vec{\jmath}\,dy$
$= (g_1'(t)\vec{\imath} + g_2'(t)\vec{\jmath})\,dt = \left(\frac{g_1'(t)}{|g'(t)|}\vec{\imath} + \frac{g_2'(t)}{|g'(t)|}\vec{\jmath}\right)|g'(t)|\,dt \overset{\text{def}}{=} \vec{t}\,ds,$
where $\vec{t}$ is the unit tangent vector to the curve, and ds is the arc-length element.

Green's theorem rewritten:

$$\int_{\substack{\text{closed}\\\text{curve}}} \vec{F} \cdot \vec{t}\,ds = \iint_{\substack{\text{region}\\\text{inside}}} (\nabla \times \vec{F}) \cdot \vec{k}\,dx\,dy.$$

The path integral is the *circulation* of $\vec{F}$ around the curve.
The quantity $\nabla \times \vec{F}$ is the *curl* of $\vec{F}$.

Divergence theorem (in the plane)

If the curve is $g(t) = (g_1(t), g_2(t))$, then the unit *normal* vector is $\vec{n} = \frac{g_2'(t)}{|g'(t)|}\vec{i} - \frac{g_1'(t)}{|g'(t)|}\vec{j}$. If we write $\vec{F} = F_1\vec{i} + F_2\vec{j}$, then $\vec{F} \cdot \vec{n}\, ds = (F_1 g_2'(t) - F_2 g_1'(t))\, dt = F_1\, dy - F_2\, dx$.

Green's theorem implies that $\int_{\text{closed curve}} \vec{F} \cdot \vec{n}\, ds =$

$$\iint_{\text{region inside}} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dx\, dy = \iint_{\text{region inside}} \nabla \cdot \vec{F}\, dx\, dy.$$

The path integral is the *flux* of $\vec{F}$ across the curve.
The quantity $\nabla \cdot \vec{F}$ is the *divergence* of $\vec{F}$.