Math 311-102

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Surface area

Example. A paraboloid is described by the parametric equation $g(u,v) = (u, v, u^2 + v^2)$. Find the area of the part of the surface for which $u^2 + v^2 \leq 1$.

Solution. Two tangent vectors are $\frac{\partial g}{\partial u} = (1, 0, 2u)$ and $\frac{\partial g}{\partial v} = (0, 1, 2v)$. The *surface area element* $d\sigma$ equals $\left|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right| \, du \, dv = \sqrt{1 + 4u^2 + 4v^2} \, du \, dv$. The surface area is
$\iint\limits_{u^2+v^2 \leq 1} \sqrt{1 + 4u^2 + 4v^2} \, du \, dv = \iint\limits_{\substack{0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi}} \sqrt{1 + 4r^2} \, r \, dr \, d\theta$
$= 2\pi \int_0^1 r\sqrt{1 + 4r^2} \, dr = \frac{2\pi}{8} \int_1^5 \sqrt{t} \, dt$ (via $t = 1 + 4r^2$)
$= \frac{2\pi}{8} \frac{2}{3} t^{3/2} \Big|_1^5 = \frac{\pi}{6}(5\sqrt{5} - 1)$.

Key formula: $d\sigma = \left|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right| \, du \, dv$

Flux across a surface

Example. Find the flux of the vector field $\vec{F}(x, y, z) = (y, -x, z)$ across the surface S of the paraboloid $g(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leq 1$.

Solution. The flux equals $\iint_S \vec{F} \cdot \vec{n} \, d\sigma$, where $\vec{n}$ denotes the unit normal vector (another notation is $\iint_S \vec{F} \cdot d\vec{S}$), or $\iint\limits_{u^2+v^2 \leq 1} \vec{F} \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right) \, du \, dv$. Since $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = (-2u, -2v, 1)$, the integrand $\vec{F} \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right)$ equals $-2uv + 2uv + u^2 + v^2$, and the integral equals $\iint\limits_{u^2+v^2 \leq 1} (u^2 + v^2) \, du \, dv = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \pi/2$.

Key formula: flux $= \iint_S \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right) \, du \, dv$