

## Topics in Applied Mathematics I

To obtain full credit, show your work.

1. Determine the dimension of the subspace of  $\mathbb{R}^3$  spanned by the three vectors  $(1, 2, 3)$ ,  $(2, 3, 4)$ , and  $(3, 4, 5)$ . Explain your reasoning.

(This is exercise 8 on page 137 of the textbook.)

The first two vectors are not multiples of each other, so they are linearly independent. Therefore the dimension of their span is 2. The question reduces to deciding whether the third vector is in the plane spanned by the first two vectors. It is, and one can see that either by inspection (the third vector is twice the second vector minus the first vector) or

by row reducing the matrix  $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$  to  $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ . Thus the original three vectors span a subspace of dimension 2.

2. Suppose  $f(\vec{x}) = A\vec{x}$  with  $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . Describe the image of the function  $f$  and the null space of the function  $f$ .

(This is exercise 14 on page 130 of the textbook.)

The image consists of linear combinations of the columns of the matrix. Since the columns are the three standard basis vectors, the image is all of  $\mathbb{R}^3$ . In other words, the image is the whole of the range space.

Because the dimension of the image in this example equals the dimension of the domain, the null space must have dimension 0. In other words, the null space consists of the single vector  $\vec{0}$ . Alternatively, one can see directly that the only vector  $\vec{x}$  satisfying  $A\vec{x} = \vec{0}$  is  $\vec{x} = \vec{0}$  because the equation  $A\vec{x} = \vec{0}$  translates into the three equations  $x_3 = 0$ ,  $x_2 = 0$ , and  $x_1 = 0$ .

3. Suppose  $F: C(-\infty, \infty) \rightarrow C(-\infty, \infty)$  (the space of continuous functions on the real numbers) is defined by  $F(u)(x) = e^{u(x)}$ . Is the transformation  $F$  linear? Explain why or why not.

(This is exercise 8 on page 130 of the textbook.)

**Topics in Applied Mathematics I**

The transformation  $F$  is not linear. To show this, it is enough either to give an example in which  $F(u + v) \neq F(u) + F(v)$  or to give an example in which  $F(au) \neq aF(u)$ .

One example for the first case could be to take  $u$  and  $v$  both equal to the constant function 1. Then  $F(u + v)$  is the constant function  $e^2$ , and  $F(u) + F(v)$  is the sum  $e^1 + e^1$ ; evidently these are not equal.

One example for the second case could be to take  $u(x) = x$  and  $a = 2$ . Then  $F(2u)$  is the function  $e^{2x}$ , and  $2F(u)$  is the function  $2e^x$ ; evidently these are not equal.

4. Suppose  $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$ . Find a basis for the null space of  $A$ . Explain your reasoning.

The second row is a multiple of the first, so the null space consists of all vectors  $(x, y, z)$  such that  $x + 2y + 4z = 0$ . This null space is a plane, and any pair of linearly independent vectors in that plane will serve as a basis. Consequently, the answer is not uniquely determined.

One natural pair of basis vectors arises by setting either  $y$  or  $z$  equal to 0 to obtain the vectors  $(4, 0, -1)$  and  $(2, -1, 0)$ .

5. Suppose  $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$  as in the preceding problem. Find a basis for the image of  $A$ . Explain your reasoning.

The image consists of all linear combinations of the columns of the matrix  $A$ . Since the second column is 2 times the first, and the third column is 4 times the first, the image consists of all scalar multiples of the vector  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . The image has dimension 1, and the vector  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is a basis for the image. (Any non-zero multiple of that vector is a correct answer too.)