Topics in Applied Mathematics I

The following problems all concern the matrix $A = \begin{pmatrix} -4 & -4 & 8 \\ 2 & 2 & -2 \\ -3 & -3 & 7 \end{pmatrix}$.

1. Find the eigenvalues of the matrix A.

   Set $0 = \det(A - \lambda I) = \begin{vmatrix} -4-\lambda & -4 & 8 \\ 2 & 2-\lambda & -2 \\ -3 & -3 & 7-\lambda \end{vmatrix}$. To simplify the computation of the determinant, subtract the second column from the first column to get $\begin{vmatrix} -\lambda & -4 & 8 \\ \lambda & 2-\lambda & -2 \\ 0 & -3 & 7-\lambda \end{vmatrix}$. Add the first row to the second row to get $\begin{vmatrix} -\lambda & -4 & 8 \\ 0 & -2-\lambda & 6 \\ 0 & -3 & 7-\lambda \end{vmatrix}$. Now expand on the first column. The characteristic equation becomes $0 = -\lambda[(-2-\lambda)(7-\lambda)+18] = -\lambda(\lambda^2 - 5\lambda + 4) = -\lambda(\lambda-1)(\lambda-4)$. Therefore the three eigenvalues are 0, 1, and 4.

2. For each eigenvalue, find a corresponding eigenvector.

   For $\lambda = 0$, solve the equation $(A - 0I)\vec{v} = 0$ by row reducing:
   $\begin{pmatrix} -4 & -4 & 8 \\ 2 & 2 & -2 \\ -3 & -3 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.
   For $\lambda = 1$, solve the equation $(A - 1I)\vec{v} = 0$ by row reducing:
   $\begin{pmatrix} -5 & -4 & 8 \\ 2 & 1 & -2 \\ -3 & -3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$, so $\vec{v} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.
   For $\lambda = 4$, solve the equation $(A - 4I)\vec{v} = 0$ by row reducing:
   $\begin{pmatrix} -8 & -4 & 8 \\ 2 & -2 & -2 \\ -3 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

3. Find a matrix U such that $U^{-1}AU$ is equal to a diagonal matrix D.

Topics in Applied Mathematics I

The matrix U should have the eigenvectors as its columns, so $U = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then $U^{-1}AU$ will be the diagonal matrix $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ with the eigenvalues on the diagonal.

You can compute that $U^{-1} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ and then verify that indeed $U^{-1}AU = D$ by carrying out the matrix multiplication.

4. Write the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ as a linear combination of the eigenvectors.

If $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = U^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Using the expression for U^{-1} written above gives $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$. Thus $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

5. Find a matrix B such that $B^2 = A$.
   Hint: Observe that $U^{-1}B^2U = (U^{-1}BU)^2$.

Since $D = U^{-1}AU = (U^{-1}BU)^2$, and since D is the square of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, we can set $U^{-1}BU = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ or, equivalently, $B = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} U^{-1}$. Computing this matrix product gives the answer $B = \begin{pmatrix} -2 & -2 & 4 \\ 2 & 2 & -2 \\ -1 & -1 & 3 \end{pmatrix}$. You can multiply out B^2 to check the answer.