Topics in Applied Mathematics I

1. Does the formula $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1$ determine an inner product on the vector space $\mathbb{R}^2$? Explain why or why not.

   (This is exercise 4 on page 158 of the textbook.)

   An inner product is supposed to have three properties: positivity, symmetry, and linearity (linearity = additivity & homogeneity). Although the formula satisfies the properties of symmetry and linearity, it fails the property of positivity. For example, the vector $(0, 1)$ is not the zero vector, yet $\langle (0, 1), (0, 1) \rangle = 0$. Thus the formula does not determine an inner product.

2. Suppose $f(t) = (t\cos(t), t\sin(t))$ (a parametric representation of a curve in the plane). Find a parametric representation for the tangent line to the curve at the point where $t = \pi/2$.

   (This is exercise 2 on page 182 of the textbook.)

   The point on the curve is $f(\pi/2)$ or $(0, \pi/2)$. The direction of the tangent line is $f'(\pi/2)$ or $(\cos(t) - t\sin(t), \sin(t) + t\cos(t))\big|_{\pi/2}$ or $(-\pi/2, 1)$. A parametric representation of the tangent line in terms of parameter s is therefore $(0, \pi/2) + s(-\pi/2, 1)$.

   The answer is not unique. One could, for example, replace the vector $(-\pi/2, 1)$ by its negative or by any scalar multiple of itself.

3. Suppose $g(u, v) = (u, v, uv)$ (a parametric representation of a surface in $\mathbb{R}^3$). Find two linearly independent vectors tangent to the surface at the point where $u = 1$ and $v = 1$.

   (This is exercise 8 on page 211 of the textbook.)

   We actually talked about this problem at the beginning of class (in response to a question).

   One tangent vector is $\frac{\partial g}{\partial u}(1, 1) = (1, 0, v)\big|_{(1,1)} = (1, 0, 1)$. Another tangent vector is $\frac{\partial g}{\partial v}(1, 1) = (0, 1, u)\big|_{(1,1)} = (0, 1, 1)$. These vectors are not scalar multiples of each other, so they are linearly independent.

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4. Suppose $f(x,y) = x^2 - y^2$ and $\vec{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $\vec{x}_0 = (2,1)$. Find the directional derivative of the function f in the direction of the unit vector $\vec{u}$ at the point $\vec{x}_0$.

   (This is exercise 2 on page 236 of the textbook.)

   We need to compute the quantity $\nabla f(\vec{x}_0) \cdot \vec{u}$.

   Now $\nabla f(\vec{x}_0) = (2x, -2y)\big|_{(2,1)} = (4,-2)$, so $\nabla f(\vec{x}_0) \cdot \vec{u} = (4,-2) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{2}{\sqrt{2}} = \sqrt{2}$.

5. Suppose $f(u,v,w) = \begin{pmatrix} uv \\ vw \\ wu \end{pmatrix}$. Find the derivative matrix f' (the Jacobian matrix).

   (This is exercise 7 on page 244 of the textbook.)

   The rows of the derivative matrix are the gradients of the component functions, so the matrix is $\begin{pmatrix} \dfrac{\partial(uv)}{\partial u} & \dfrac{\partial(uv)}{\partial v} & \dfrac{\partial(uv)}{\partial w} \\ \dfrac{\partial(vw)}{\partial u} & \dfrac{\partial(vw)}{\partial v} & \dfrac{\partial(vw)}{\partial w} \\ \dfrac{\partial(wu)}{\partial u} & \dfrac{\partial(wu)}{\partial v} & \dfrac{\partial(wu)}{\partial w} \end{pmatrix} = \begin{pmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{pmatrix}$.