Topics in Applied Mathematics I

1. Let γ be the path consisting of line segments in the plane from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(1,2)$, and from $(1,2)$ back to $(0,0)$. Evaluate the integral
$$\int_\gamma (-xy + \sin(x^2))\,dx + (\cos^2 y)\,dy.$$
(This is exercise 2 on page 457 in the textbook.)

By Green's theorem, the integral equals $\iint_T x\,dx\,dy$, where T is the triangle with the indicated vertices. The hypotenuse of the triangle is part of the line $y = 2x$, so this area integral equals $\int_0^1 x\left(\int_0^{2x} dy\right) dx = \int_0^1 2x^2\,dx = 2/3$.

2. Find a function f such that $\nabla f = (3x^2y, x^3 + 3y^2)$.
(This is exercise 1(a) on page 457 in the textbook.)

Since $\frac{\partial f}{\partial x} = 3x^2y$, there must be a function $g(y)$ such that $f(x,y) = x^3y + g(y)$. Then $x^3 + 3y^2 = \frac{\partial f}{\partial y} = x^3 + g'(y)$, so $g(y) = y^3 + c$ for some constant c. Thus $f(x,y) = x^3y + y^3 + c$.

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3. If C is a simple closed curve in the plane bounding a region R, which one of the following integrals is certain to be equal to 0? Explain why.

   (a) $\int_C x\,dx$ (b) $\int_C y\,dx$ (c) $\int_C x\,dy$ (d) $\int_C xy\,dy$

   By Green's theorem, the first integral equals $\iint_R 0\,dx\,dy$, so it is certain to be equal to 0.

   The second integral equals the negative of the area of R, so it is never equal to 0.

   The third integral equals the area of R, so it is never equal to 0.

   The fourth integral equals $\iint_R y\,dx\,dy$, which may or may not equal 0, depending on the region R.

4. Let $\vec{G}(x, y, z) = (y, z, x)$. Is $\vec{G}$ a gradient field in $\mathbb{R}^3$? Explain why or why not.

   (This is exercise 4 on page 418 in the textbook.)

   The derivative matrix of $\vec{G}$ equals $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. This matrix is not symmetric, so $\vec{G}$ is not a gradient field.

5. Suppose γ_1 and γ_2 are two paths in the plane joining the points $(0, 0)$ and $(2, 3)$. Must $\int_{\gamma_1} y\,dx + x\,dy$ be equal to $\int_{\gamma_2} y\,dx + x\,dy$? Explain why or why not.

   (This is a variation of exercise 5 on page 408 in the textbook.)

   If $f(x, y) = xy$, then $\nabla f \cdot d\vec{x} = y\,dx + x\,dy$. Consequently, the value of the integral $\int_\gamma y\,dx + x\,dy = f(2, 3) - f(0, 0) = 6$. The answer is independent of the path γ, as long as γ starts at $(0, 0)$ and ends at $(2, 3)$.