Exam 2, March 20

1. Solve the heat equation
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \quad 0 < t < \infty,$$
on the semi-infinite interval $(0, \infty)$, where the boundary condition is $u(0,t) = 0$, and the initial condition is given by the piecewise-defined function
$$u(x,0) = \begin{cases} \sin x, & \text{if } 0 \le x \le \pi \\ 0, & \text{if } \pi \le x. \end{cases}$$
You may leave one unevaluated Fourier integral in the answer.

2. The acoustical vibration of air in a column, as in an organ pipe, may be modeled by the wave equation
$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}, \qquad 0 < x < \pi, \quad 0 < t < \infty,$$
where $p(x,t)$ denotes the air pressure at position x and time t.

   Suppose that both ends of the pipe are open, so that the air pressure at the ends is always equal to the air pressure p_0 of the environment. In other words, the boundary conditions are $p(0,t) = p_0$ and $p(\pi,t) = p_0$. (Notice that these boundary conditions are not homogeneous ones.)

   Suppose that at time $t = 0$, the organ player sets the air into motion according to the initial conditions
$$p(x,0) = p_0 \quad \text{and} \quad \frac{\partial p}{\partial t}(x,0) = \cos x \qquad \text{when } 0 < x < \pi.$$
Use **either** d'Alembert's method **or** the method of separation of variables to find the solution for $p(x,t)$.

You may wish to use the following trigonometric identities:
$$(\sin A)(\sin B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$(\sin A)(\cos B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$
$$(\cos A)(\cos B) = \frac{1}{2}(\cos(A-B) + \cos(A+B)).$$