Exam 3, April 15

1. Solve the following boundary value problem for the potential equation (Laplace's equation) on a square:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{for} \quad 0 < x < \pi \quad \text{and} \quad 0 < y < \pi,$$

with the boundary conditions

$$u(0,y) = 1 \quad \text{and} \quad \frac{\partial u}{\partial x}(\pi, y) = 0 \quad \text{when} \quad 0 < y < \pi,$$
$$u(x,0) = 1 \quad \text{and} \quad \frac{\partial u}{\partial y}(x, \pi) = 0 \quad \text{when} \quad 0 < x < \pi.$$

2. Do **one** of the following two problems.

(a) Recall that the Legendre polynomial $P_n(x)$ is a polynomial solution of degree n of the differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

normalized by the condition $y(1) = 1$. Compute $P_4(x)$.

(b) The Hermite polynomial $H_n(x)$ is a polynomial solution of degree n of the differential equation

$$y'' - 2xy' + 2ny = 0,$$

where n is a positive integer. Show that the Hermite polynomials satisfy the weighted orthogonality relation

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2}\, dx = 0$$

when $n \neq m$.
Hint: observe that $\left(e^{-x^2} y'\right)' = (y'' - 2xy')e^{-x^2}$.