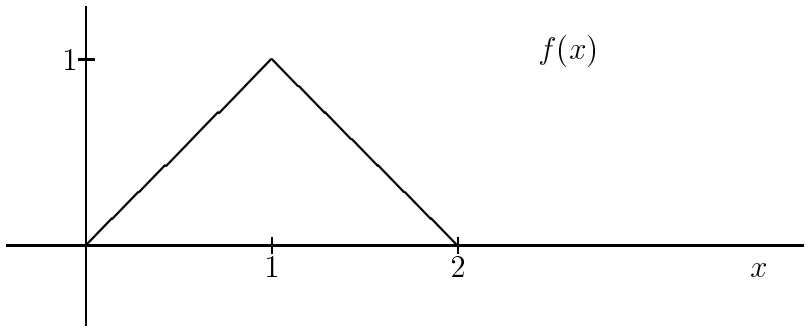
Music in Wonderland

When Alice returned later to the Mad Tea-Party, she found the March Hare picking out a tune on the most curious guitar she had ever seen. The neck of the guitar stretched out across the table, past the house, and on into the distance as far as she could see.

As Alice sat down next to the Dormouse, who was fast asleep, the Hatter reached over her shoulder (which Alice thought was very rude) and pressed one finger down on a guitar string while stretching the string upward with the other hand (bringing the string into the position shown in the diagram).



"Shall I beat time?" asked the March Hare.

"Let's beat the Dormouse," replied the Hatter, raising his hands threateningly over his head.

"Oh, no, don't do that!" exclaimed Alice in dismay, but she was suddenly distracted as her glance fell upon the guitar string, which was now vibrating in a most curious manner.

1. Without doing any calculations, make a guess about how the wave propagates along the infinitely long string.

2. Modeling the situation by the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ for $x > 0$ and $t > 0$ with boundary condition $u(0,t) = 0$ and initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$, apply d'Alembert's method to find a solution in the form $u(x,t) = \psi(x+ct) + \phi(x-ct)$.
(You will need to make an appropriate extension of the function f to the whole real line.)

3. Use your solution to sketch the graph of $u(x,t)$ as a function of x when $t = 1/(2c)$, $t = 1/c$, $t = 2/c$, $t = 3/c$. How good was your guess in part 1?

4. Solve the problem a second time, now using the method of separation of variables, to derive the integral representation

$$u(x,t) = \frac{4}{\pi} \int_0^\infty \frac{(\sin(\lambda))(1-\cos(\lambda))}{\lambda^2} (\sin(\lambda x))(\cos(\lambda ct))\, d\lambda.$$