# Linear Algebra

**Instructions** Please answer the five problems on your own paper. These are essay questions: you should write in complete sentences.

1. Jordan is using a TI-89 calculator to help analyze the linear system $A\mathbf{x} = \mathbf{b}$, where $A$ is a certain $3 \times 4$ matrix and $\mathbf{b}$ is a certain $3 \times 1$ matrix (a column vector). Jordan applies the `rref` command to the augmented coefficient matrix and obtains the result

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(Jordan's calculator does not show a vertical bar to separate the last column from the coefficient matrix.) Discuss what information Jordan can deduce about the original linear system.
[For instance, is the system underdetermined or overdetermined? consistent or inconsistent? Is there a unique solution? Does Jordan have enough information to write down the solution(s)?]

**Solution.** The indicated reduced row echelon form reveals that we have a consistent, underdetermined system with lead variables $x_1$ and $x_3$ and free variables $x_2$ and $x_4$. There are infinitely many solutions, and we can exhibit the solutions as follows:

$$\mathbf{x} = \begin{bmatrix} 4 - 2\beta \\ \alpha \\ -5 - 3\beta \\ \beta \end{bmatrix}, \qquad \text{where } \alpha \text{ and } \beta \text{ take arbitrary values.}$$

2. Consider the system of three simultaneous equations

$$\begin{cases} x_1 + \ \ x_2 \qquad\quad = 2 \\ ax_1 + ax_2 \qquad\quad = 3a \\ bx_1 + bx_2 + ax_3 = 4 + b \end{cases}$$

for the unknowns $x_1$, $x_2$, and $x_3$. Find all values of $a$ and $b$ for which this system of equations is consistent.
Explain your reasoning in complete sentences.

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**Solution.** Method 1 (using rows). The most popular approach was to bring the augmented coefficient matrix to row echelon form as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ a & a & 0 & 3a \\ b & b & a & 4+b \end{array}\right] \xrightarrow[R3 \to R3 - bR1]{R2 \to R2 - aR1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 4-b \end{array}\right]$$
$$\xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & a & 4-b \\ 0 & 0 & 0 & a \end{array}\right].$$

The bottom row of this echelon form shows that a necessary condition for the system to be consistent is that $a = 0$. Inserting this condition into the second row shows that another necessary condition for consistency is that $b = 4$.

When both of these necessary conditions ($a = 0$ and $b = 4$) hold, the echelon form reduces to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which evidently represents a consistent system. Indeed, we can exhibit the solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2-\alpha \\ \alpha \\ \beta \end{bmatrix}, \qquad \text{where } \alpha \text{ and } \beta \text{ take arbitrary values.}$$

Thus the two conditions $a = 0$ and $b = 4$ together are necessary and sufficient for the system to be consistent.

Method 2 (using columns). By the consistency theorem for linear systems (Theorem 1.3.1), we know that the system is consistent if the column vector on the right-hand side is a linear combination of the columns of the coefficient matrix. The first two columns of the coefficient matrix are identical, so we want to know if there exist constants $c_1$ and $c_2$ for which

$$c_1 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} 2 \\ 3a \\ 4+b \end{bmatrix}.$$

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Looking at the first component of the column vectors, we see that the only possible choice for $c_1$ is 2. Then looking at the second component of the column vectors, we find that $2a = 3a$, which means that $a = 0$. Looking now at the third component of the column vectors, we see that $2b = 4 + b$, which means that $b = 4$. In conclusion, we can write the column vector on the right-hand side as a linear combination of the columns of the coefficient matrix if and only if both $a = 0$ and $b = 4$.

**Remark** The TI-89 calculator will answer this problem incorrectly, unless you are very careful about what question you ask it. If you apply the `rref` command to the augmented coefficient matrix, then the calculator returns a matrix with bottom row [0 0 0 1], falsely indicating that the system is always inconsistent. (The calculator silently divides by $a$, ignoring the special case when $a = 0$.)

3. Suppose

$$A = \begin{bmatrix} 0 & a & 1 \\ 1 & 0 & 1 \\ 0 & 0 & a \end{bmatrix}.$$

Determine the value(s) of $a$ for which the matrix $A$ is invertible.
[If you do a computation to solve this problem, say what computation you are doing and why.]

**Solution.** We know that a square matrix is invertible precisely when its determinant is non-zero. Therefore we can answer the question by computing $\det(A)$. A cofactor expansion using the bottom row shows that

$$\det(A) = a \det \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} = -a^2.$$

Consequently, the matrix $A$ is invertible precisely when $a \neq 0$.

4. Suppose that $A$ is an $n \times n$ matrix, and $S$ is an invertible $n \times n$ matrix. Show that $\det(S^{-1}AS) = \det(A)$.

**Solution.** We know that the determinant is a multiplicative function: the determinant of a product equals the product of the determinants. (See Theorem 2.2.3.) Therefore

$$\det(S^{-1}AS) = \det(S^{-1}) \det(A) \det(S).$$

We also know that the product $\det(S^{-1})\det(S) = 1$. (This property was a homework problem: exercise 6 on page 104. The proof is the following calculation using the multiplicative property: $1 = \det(I) = \det(S^{-1}S) = \det(S^{-1})\det(S)$.) Therefore the right-hand side of the displayed equation simplifies to $\det(A)$, which is the required result.

There is a subtle point here. Matrix multiplication is not commutative, so we cannot simplify $S^{-1}AS$ to get $A$. But multiplication of *numbers* is commutative, so we *can* simplify $\det(S)^{-1}\det(A)\det(S)$ to get $\det(A)$.

5. Maude is studying the set of all polynomials in $x$ of odd degree. Help Maude decide if this *set* forms a *vector space* (under the usual operations of addition and scalar multiplication).

**Solution.** Since the operations are the usual ones, the validity of the commutative, associative, and distributive laws is not in question. The set fails to be a vector space, however, for two reasons.

(1) There is no additive identity element in the set. The only candidate for the additive identity is the zero polynomial, which is not a polynomial of odd degree.

(2) The addition operation is not well defined; that is, the sum of two elements of the set need not be in the set. For instance, the polynomial $x^3 + x^2$ has odd degree (namely, degree 3), and the polynomial $-x^3 + x$ has odd degree, but their sum $x^2 + x$ has even degree and hence is not an element of the set. (We did an example like this in class.) In other words, our set is not *closed* under the addition operation.