

Linear Algebra

Instructions Please answer the six problems on your own paper. These are essay questions: you should write in complete sentences.

1. Are the two matrices

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ 5 & 2 & 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & -2 \\ 5 & 5 & 2 \end{bmatrix}$$

row equivalent? Explain why or why not.

[Recall that two matrices are row equivalent if one matrix can be converted into the other by a sequence of elementary row operations.]

Solution. The two matrices are row equivalent if and only if they have the same reduced row echelon form. Here is the computation of the reduced row echelon form for the first matrix:

$$\begin{aligned} \begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ 5 & 2 & 7 \end{bmatrix} &\xrightarrow[\substack{R2 \rightarrow R2 - 2R1 \\ R3 \rightarrow R3 - 5R1}]{R2 \rightarrow R2 - 2R1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 12 & 12 \end{bmatrix} \xrightarrow[\substack{R2 \div 5 \\ R3 \div 12}]{R2 \div 5} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &\xrightarrow[\substack{R3 \rightarrow R3 - R1 \\ R1 \rightarrow R1 + 2R2}]{R3 \rightarrow R3 - R1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Here is the computation of the reduced row echelon form for the second matrix:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & -2 \\ 5 & 5 & 2 \end{bmatrix} &\xrightarrow[\substack{R2 \rightarrow R2 - 4R1 \\ R3 \rightarrow R3 - 5R1}]{R2 \rightarrow R2 - 4R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -6 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[\substack{R2 \div -6 \\ R3 \div -3}]{R2 \div -6} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &\xrightarrow[\substack{R3 \rightarrow R3 - R2 \\ R1 \rightarrow R1 - R2}]{R3 \rightarrow R3 - R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The two reduced row echelon forms are different, so the two matrices are not row equivalent.

An alternative solution is to observe that the second matrix has its first two columns equal. Since row operations preserve column dependencies, any row equivalent matrix must have its first two columns equal. Since the first matrix lacks this property, the first matrix is not row equivalent to the second matrix.

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2. Determine all values of t for which the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3t & 4 & 0 & 0 \\ 0 & 5t & 6 & 0 \\ 7 & 0 & 0 & 8t \end{bmatrix}$$

is singular (that is, not invertible).

Solution. A square matrix is singular if and only if the determinant is equal to 0. The determinant can be computed conveniently by using a cofactor expansion on the last column:

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3t & 4 & 0 & 0 \\ 0 & 5t & 6 & 0 \\ 7 & 0 & 0 & 8t \end{vmatrix} = 8t \begin{vmatrix} 1 & 2 & 0 \\ 3t & 4 & 0 \\ 0 & 5t & 6 \end{vmatrix} = 48t \begin{vmatrix} 1 & 2 \\ 3t & 4 \end{vmatrix} = 48t(4 - 6t).$$

The determinant is equal to 0 when $t = 0$ and when $t = 2/3$, so these are the values of t for which the matrix is singular.

3. Create a 3×3 matrix from your nine-digit student identification number by putting the first three digits in the first row of the matrix, the next three digits in the second row of the matrix, and the final three digits in the third row of the matrix. Determine the dimension of the column space of your matrix. Explain your method.

Solution. The dimension of the column space is equal to the dimension of the row space. One way to proceed is to reduce your matrix to echelon form and see how many leading 1's there are.

Three vectors in R^3 chosen at random are most likely linearly independent, so most students have a matrix of rank equal to 3. In that case, the matrix is invertible, so you could detect this situation also by checking to see if the determinant of the matrix is nonzero.

4. In the vector space $C[0, 1]$ of continuous functions, the set of functions $[1, x, \sin(x), \cos(x)]$ forms a basis B for a certain four-dimensional subspace W . Find the 4×4 matrix that represents, with respect to the basis B , the linear operator D on W defined by differentiation. [That

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is, $D(1) = 0$, $D(x) = 1$, $D(\sin(x)) = \cos(x)$, and $D(\cos(x)) = -\sin(x)$.] Explain your reasoning.

Solution. The first column of the matrix specifies the coordinates in the basis B of the image of the first basis vector, the second column specifies the coordinates in the basis B of the second basis vector, and so on. Therefore the matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

5. Find the line of the form $y = c_1 + c_2x$ that gives the best least-squares fit to the following data:

$$\begin{array}{c|c|c|c} x & 0 & 1 & 2 \\ \hline y & 1 & 3 & -1 \end{array}$$

Solution. Here is the corresponding linear system of equations:

$$\begin{array}{l} 1 = c_1 + c_2 \cdot 0 \\ 3 = c_1 + c_2 \cdot 1 \\ -1 = c_1 + c_2 \cdot 2 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

It is easy to see that this overdetermined system is inconsistent. To set up the corresponding least-squares problem, we multiply both sides by the transpose of the coefficient matrix to obtain the following new system:

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Now solve by your favorite method. For instance, multiply by the inverse matrix to find that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Therefore the line that gives the best least-squares fit to the data has the equation $y = 2 - x$.

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6. Find a diagonal matrix that is similar to the matrix

$$\begin{bmatrix} 3 & 2 & 3 \\ 5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

[Recall that two square matrices are similar if they represent the same linear operator with respect to two different bases. Equivalently, A and B are similar matrices if $B = S^{-1}AS$ for some invertible matrix S .] Explain your method.

Solution. We start by determining the eigenvalues from the characteristic equation:

$$\begin{aligned} 0 &= \begin{vmatrix} 3-\lambda & 2 & 3 \\ 5 & 0-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 3-\lambda & 2 \\ 5 & -\lambda \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 - 3\lambda - 10) = (1-\lambda)(\lambda-5)(\lambda+2). \end{aligned}$$

Therefore the eigenvalues are 1, 5, and -2 .

The eigenvectors corresponding to these three eigenvalues are linearly independent and so form a basis for R^3 . In that basis, the corresponding linear operator is represented by a diagonal matrix with the eigenvalues on the diagonal. That is the matrix we seek.

There are six correct answers, corresponding to the six possible ways of ordering the diagonal entries. One of these answers is

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Notice that it is not necessary to compute the eigenvectors to answer this question.

Bonus (up to 10 points extra credit)

The so-called *Legendre polynomials* are an orthogonal set of functions with respect to the inner product

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x) dx.$$

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The Legendre polynomial $P_n(x)$ is a polynomial of degree n that is normalized not by the condition $\|P_n\| = 1$ but instead by the condition that $P_n(1) = 1$. Here are the first three Legendre polynomials:

$$\begin{aligned}P_0(x) &= 1 \\P_1(x) &= x \\P_2(x) &= \frac{1}{2}(3x^2 - 1).\end{aligned}$$

Determine the degree-three polynomial $P_3(x)$.

Solution. One method is to write $P_3(x)$ in the form $a + bx + cx^2 + dx^3$, write three equations that express the orthogonality of P_3 with P_0 , P_1 , and P_2 , write a fourth equation that says $P_3(1) = 1$, and solve the four simultaneous equations for the coefficients a , b , c , and d .

A somewhat simpler method is to start with x^3 and use the Gram-Schmidt process to subtract off its projection on the subspace spanned by P_0 , P_1 , and P_2 . Since these three polynomials are given to be orthogonal, this projection is

$$\frac{\langle x^3, P_0 \rangle}{\|P_0\|^2} P_0 + \frac{\langle x^3, P_1 \rangle}{\|P_1\|^2} P_1 + \frac{\langle x^3, P_2 \rangle}{\|P_2\|^2} P_2.$$

The inner product is integration over a symmetric interval, so $\langle x^3, P_0 \rangle = 0$ and $\langle x^3, P_2 \rangle = 0$ by symmetry. Moreover

$$\langle x^3, P_1 \rangle = \int_{-1}^1 x^4 dx = \frac{2}{5}, \quad \text{and} \quad \|P_1\|^2 = \int_{-1}^1 x^2 dx = \frac{2}{3}.$$

Therefore the function $x^3 - \frac{3}{5}x$ (along with any scalar multiple of it) is orthogonal to the subspace spanned by P_0 , P_1 , and P_2 . This function takes the value $\frac{2}{5}$ when $x = 1$, so dividing it by that value normalizes it according to the given scheme. Therefore $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.