Complex Variables

Instructions Solve any **eight** of the following ten problems. Explain your reasoning in complete sentences to maximize credit.

1. The TI-89 calculator says, reasonably enough, that

$$\lim_{x\to 0}\left((x-1)^{1/3}-1\right)^3 = -8.$$

Somewhat surprisingly, Maple and Mathematica say instead that

$$\lim_{x\to 0}\left((x-1)^{1/3}-1\right)^3 = 1.$$

Use complex numbers to explain how the two different answers both can be justified mathematically.

Remark The Maple command is `limit(((x-1)^(1/3)-1)^3,x=0)`, and the Mathematica command is `Limit[((x-1)^(1/3)-1)^3,x->0]`.

2. You know very well that

$$\sin^2(x) + \cos^2(x) = 1 \qquad \text{for every real number } x.$$

Prove that

$$\sin^2(z) + \cos^2(z) = 1 \qquad \text{for every complex number } z.$$

3. Do **either** part (a) **or** part (b).

   (a) Determine a (non-closed) path γ in the complex plane such that

$$\int_\gamma (2z+1)\,dz = -1.$$

   (b) The value of the line integral $\displaystyle\int_\gamma \frac{1}{z^2(z^2+1)}\,dz$ depends on γ, the integration path. What are the possible values of this integral as γ varies over all simple closed curves?

4. Find an entire function $f(z)$ whose real part $u(x,y)$ equals $x^2 - y^2 - 2y$ (where, as usual, x and y denote the real part and the imaginary part of the complex variable z).

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5. Give an example of a power series $\sum_{n=0}^{\infty} a_n z^n$ that has radius of convergence equal to 3 and that represents an analytic function having no zeroes.

6. Evaluate the integral

$$\int_{|z|=1} z^{407} \cos(1/z)\, dz,$$

where the integration curve is the unit circle with its usual counterclockwise orientation. (Recall that $\sum_{n=0}^{\infty}(-1)^n w^{2n}/(2n)! = \cos(w)$.)

7. How many solutions are there to the equation

$$z^4 + 4 = e^{-z}$$

in the right-hand half-plane where $\mathrm{Re}(z) > 0$? How do you know?

8. Do **either** part (a) **or** part (b).

   (a) Either find a one-to-one conformal mapping from the punctured disc $\{\, z : 0 < |z| < 1 \,\}$ onto the annulus $\{\, z : 1 < |z| < 2 \,\}$ or prove that none exists.

   (b) Either find a one-to-one conformal mapping from the first quadrant $\{\, z : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0 \,\}$ onto the strip $\{\, z : |\,\mathrm{Im}(z)| < 1 \,\}$ or prove that none exists.

9. For the function $\dfrac{1+z}{z(1-z)}$, find a Laurent series in powers of z and $\dfrac{1}{z}$ that converges when $0 < |z| < 1$.

10. The TI-89 calculator, Maple, and Mathematica all agree that

$$\int_0^{\infty} \frac{x^2}{x^4+1}\, dx = \frac{\pi\sqrt{2}}{4}.$$

Use contour integration and residues to prove this formula.