Complex Variables

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Determine the polar representation of the complex number $1 + i$.

   Solution. The modulus of $1 + i$ equals $\sqrt{1^2 + 1^2}$ or $\sqrt{2}$, and the argument is $\pi/4$, so the polar representation is

   $$\sqrt{2}\,(\cos(\pi/4) + i\sin(\pi/4)).$$

   One could just as well replace the angle $\pi/4$ by $9\pi/4$ or by $\pi/4$ plus any integral multiple of 2π.

2. Suppose z is a complex number such that $|z| = 2$ and $\arg z = -3\pi/2$. Express z in its standard form $x + iy$.

   Solution. The polar form of z is $2(\cos(-3\pi/2) + i\sin(-3\pi/2))$. Since $-3\pi/2$ represents the same position as angle $\pi/2$, and $\cos(\pi/2) = 0$, while $\sin(\pi/2) = 1$, we have $z = 2i$.

3. Every complex number z has the property that $|\operatorname{Re} z| \le |z|$. Why?

   Solution. The algebraic explanation is that

   $$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \ge \sqrt{(\operatorname{Re} z)^2} = |\operatorname{Re} z|.$$

   Equality holds in the inequality precisely when $\operatorname{Im} z = 0$, that is, when z is a real number.

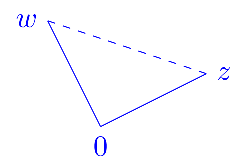
   The geometric explanation is that $|z|$ represents the length of the hypotenuse of a right triangle, while $|\operatorname{Re} z|$ represents the length of one side of the triangle. The hypotenuse is longer than either of the other sides (unless the triangle degenerates into a line segment, in which case equality can hold).

Complex Variables

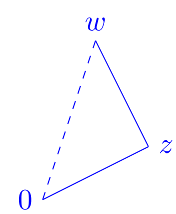
4. Suppose the complex numbers 0, z, and w represent the vertices of an isosceles right triangle. If $z = 2 + i$, find a corresponding value for w. [The answer is not unique.]

Solution. One could solve the problem geometrically in real x and y coordinates and then translate the answer into complex form. Here is a solution using the notation of complex numbers. Suppose $w = a + ib$, where a and b are real numbers. The strategy is to write two equations for the two unknowns a and b, and then solve for the values of a and b.

If the right angle is at 0, then z and w represent orthogonal vectors, so $\operatorname{Re}(z\overline{w}) = 0$ (see page 8 of the textbook). Therefore $2a + b = 0$, or $b = -2a$. Since the triangle is isosceles, $|z|^2 = |w|^2$, or $2^2 + 1^2 = a^2 + b^2$. Substituting $-2a$ for b shows that $5 = 5a^2$, or $a = \pm 1$. We get two solutions for a and b: namely, $a = 1$ and $b = -2$; and $a = -1$ and $b = 2$. The corresponding values of w are $1 - 2i$ and $-1 + 2i$.



If the right angle is at z, then $\operatorname{Re}[(w - z)\overline{z}] = 0$, that is, $\operatorname{Re} w\overline{z} = |z|^2$, or $2a + b = 5$. For the second equation, we could use that $|z| = |w - z|$, but it is simpler to observe that $|w|$ represents the hypotenuse of the isosceles right triangle, so $|w| = \sqrt{2}\,|z|$. Therefore $a^2 + b^2 = 2 \times 5 = 10$. The solutions of the two equations for a and b are $a = 3$ and $b = -1$; and $a = 1$ and $b = 3$. The corresponding values of w are $3 - i$ and $1 + 3i$.



The remaining possibility is that the right angle is at w. Then $|w| = |z|/\sqrt{2}$, or $a^2 + b^2 = 5/2$. Also $|w| = |w - z|$, or $|w|^2 = |w|^2 - 2\operatorname{Re} w\overline{z} + |z|^2$, or $2(2a + b) = 5$. Solving the pair of equations for a and b gives the solutions $\frac{3}{2} - \frac{1}{2}i$ and $\frac{1}{2} + \frac{3}{2}i$ for w.