Complex Variables

Instructions Please write your name in the upper right-hand corner of the page. Circle the correct answer. No explanation is required.

1. The set of complex numbers z such that $|z-i|^2 = 4$ represents a circle in the plane. `True` `False`

   Solution. True. The equation represents a circle with center at the point i and with radius 2.

2. The inequality $|z| + |w| \leq |z + w|$ holds for all complex numbers z and w. `True` `False`

   Solution. False. For example, if $z = 1$ and $w = -1$, then the left-hand side of the inequality equals 2, but the right-hand side equals 0.

   What is true is that $|z + w| \leq |z| + |w|$ (the triangle inequality).

3. There are exactly five complex numbers z such that $z^5 = 7 - 2i$. `True` `False`

   Solution. True. Every complex number (except for 0) has exactly five fifth roots. If the number has the polar representation $re^{i\theta}$, then the fifth roots are the products of $r^{1/5}e^{i\theta/5}$ with 1, $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, and $e^{8\pi i/5}$.

4. The set of complex numbers z such that $\operatorname{Re}(z^2) = 0$ represents a vertical line in the plane. `True` `False`

   Solution. False. If $z = x + iy$, then $\operatorname{Re}(z^2) = x^2 - y^2$. Therefore $\operatorname{Re}(z^2) = 0$ if and only if $x^2 - y^2 = 0$. The locus of points satisfying the last equation is the pair of lines $y = \pm x$.

5. An open disc in the plane is a connected set. `True` `False`

   Solution. True. Every two points in an open disc can be joined by a line segment that remains inside the disc. An open disc is a convex set, and this property is stronger than connectedness.

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6. The set of complex numbers z such that $\text{Re}(z) \geq 0$ is a closed set.
`True` `False`

   Solution. True. The boundary points of this set are the points for which $\text{Re}(z) = 0$, and all the boundary points are contained in the set.

7. The function $f(z) = \overline{z}$ is a continuous function. `True` `False`

   Solution. True: $\lim_{z \to z_0} \overline{z} = \overline{z_0}$.

8. $\displaystyle\lim_{n\to\infty} \frac{1}{(1+i)^n} = 0.$ `True` `False`

   Solution. True. The term $1 + i$ in the denominator has modulus $\sqrt{2}$, which is greater than 1, so the denominator $(1 + i)^n$ has modulus that grows without bound as n increases. Therefore the fraction tends to 0.

9. There is no complex number z for which $e^z = 0$. `True` `False`

   Solution. True: this statement is a fundamental property of the exponential function. If $z = x + iy$, then $e^z = e^x e^{iy}$, and $e^x > 0$, while e^{iy} is a complex number of modulus 1. Hence e^z is a product of two non-zero quantities.

10. The inequality $|\sin(z)| \leq 1$ holds for every complex number z.
`True` `False`

    Solution. False. We saw in class that $|\sin(z)|$ blows up as z moves up the imaginary axis. Explicitly, $\sin(iy) = \frac{i}{2}(e^y - e^{-y})$, so $|\sin(iy)| \to \infty$ as $y \to \infty$.