Final Examination

Instructions Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Prove that if z is a complex number, then $|z| = 1$ if and only if $|2z - 1| = |2 - z|$.

   Solution. Method 1. Since the absolute value is a nonnegative quantity, squaring both sides of the equations does not change the problem. Now

   $$|2z-1|^2 = |2-z|^2 \qquad \text{if and only if} \qquad (2z-1)(2\overline{z}-1) = (2-z)(2-\overline{z}),$$

   or equivalently, $4|z|^2 - 4\operatorname{Re}(z) + 1 = 4 - 4\operatorname{Re}(z) + |z|^2$. Adding $4\operatorname{Re}(z)$ to both sides and subtracting $1 + |z|^2$ from both sides produces another equivalent statement: namely,

   $$3|z|^2 = 3, \qquad \text{or} \qquad |z|^2 = 1.$$

   Thus the two given equations are algebraically equivalent.

   Method 2. A geometric restatement of the problem is to show that the fractional linear transformation $\dfrac{2z-1}{2-z}$ maps the unit circle to itself. The coefficients of this transformation are real, so the (extended) real axis maps to itself. By conformality, the unit circle maps to *some* circle that intersects the real axis orthogonally. Inspection reveals that the transformation fixes the points ± 1, so the image of the unit circle must be the unit circle.

2. Suppose a and b are real numbers, and

   $$f(x+iy) = -3x^2y + ay^3 + 2x^2 + by^2 + iv(x,y)$$

   for some real-valued function $v(x,y)$. Determine the values that a and b must have in order for f to be an analytic function.

   Solution. Method 1. For the function f to be analytic, the real part must be harmonic. Applying the Laplace operator to the given real part shows that

   $$-6y + 6ay + 4 + 2b = 0, \qquad \text{or} \qquad 6y(a-1) = -(4+2b).$$

   If this equation is to hold (on some open set), then constancy of the right-hand side implies that a must be equal to 1, for otherwise the left-hand side would not be constant. When $a = 1$, the left-hand side equals 0, so the right-hand side must equal 0 too, whence $b = -2$.

   Method 2. Apply the Cauchy–Riemann equations on some open set where f is analytic. Let $u(x,y)$ denote the real part of the function. Then

   $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -6xy + 4x, \qquad \text{so} \qquad v(x,y) = -3xy^2 + 4xy + g(x)$$

for some function g of one variable. Consequently,

$$-3y^2 + 4y + g'(x) = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3x^2 - 3ay^2 - 2by.$$

Equivalently,

$$3y^2(a-1) + y(4+2b) = 3x^2 - g'(x).$$

The left-hand side is independent of x, and the right-hand side is independent of y, so both sides must be constant functions. Being a constant polynomial of y, the left-hand side must have all its coefficients equal to 0, so $a = 1$ and $b = -2$.

3. Give precise statements of *two* of the following four items.
   - Cauchy–Riemann equations
   - Green's theorem
   - Cauchy's theorem
   - Liouville's theorem

   Solution. See equation (3.3) in Chapter II, equation (1.4) in Chapter III, the statement at the bottom of page 110 in Chapter IV, and the statement in the middle of page 118 in Chapter IV.

4. Give a concrete example of a power series $\sum_{n=1}^{\infty} a_n z^n$ for which the radius of convergence is equal to 4.

   Solution. The simplest solution is to take a_n equal to 4^{-n}. The series then becomes the geometric series $\sum_{n=1}^{\infty} (z/4)^n$, which converges precisely when $|z/4| < 1$. Thus the radius of convergence (the cut-off radius) is equal to 4.

5. Suppose the rational function

$$\frac{z}{(z-1)(z-2)}$$

   is expanded in a Laurent series in powers of z and z^{-1} that converges when $1 < |z| < 2$. Determine the coefficient of z^{407} in the series.

   Solution. Decompose the function in partial fractions:

$$\frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{2}{z-2}.$$

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(The constants in the numerators on the right-hand side are the residues of the rational function at the respective simple poles.) The term $-1/(z-1)$ is analytic outside the inner circle and therefore expands in a series in powers of z^{-1}. There is no need to compute this series, for no positive power of z appears in it. The term $2/(z-2)$ is analytic inside the outer circle and therefore expands in a series in powers of z. The answer to the problem can be obtained from this series alone.

By the geometric-series trick,

$$\frac{2}{z-2} = \frac{-1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n.$$

Accordingly, the coefficient of z^{407} equals $-1/2^{407}$.

6. Evaluate the integral

$$\int_{|z-1|=2} \frac{z}{(z^2-4)\sin(z)}\,dz,$$

where the integration path is a circle with center 1 and radius 2, oriented counterclockwise (as usual).

Solution. The apparent singularity when $z = 0$ is actually removable, since $z/\sin(z)$ has a finite limit when $z \to 0$ (namely, limit 1). Which other zeroes of the denominator are inside the curve?

The sine function additionally has zeroes at nonzero integer multiples of π, but none of these lie inside the circle, which intersects the horizontal axis at 3 (which is less than π) and at -1 (which is greater than $-\pi$). The other factor in the denominator equals 0 when $z = \pm 2$, but only the point 2 lies inside the circle.

By the residue theorem, the value of the integral can be computed as follows:

$$2\pi i \operatorname{Res}\left(\frac{z/\sin(z)}{z^2-4}, 2\right) = 2\pi i \frac{z/\sin(z)}{\frac{d}{dz}(z^2-4)}\bigg|_{z=2} = \frac{\pi i}{\sin(2)}.$$

The final answer does not simplify further.

Extra Credit

Find a conformal mapping that maps $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane) onto $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk) and takes the point 1 to the point 0.

Solution. A point z lies in the right-hand half-plane when z is closer to the point 1 than to the point -1, that is, when $|z-1| < |z+1|$. In other words, $\operatorname{Re}(z) > 0$ precisely when $\left|\frac{z-1}{z+1}\right| < 1$.

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Accordingly, the fractional linear transformation (Möbius transformation) $\dfrac{z-1}{z+1}$ maps the right-hand half-plane to the unit disk, and evidently the point 1 maps to 0. You know from class on September 27 that Möbius transformations are one-to-one and conformal, so this transformation satisfies the required properties.

The answer is not unique, for the transformation could be composed with an arbitrary rotation about the origin. Multiplying the transformation by -1 is actually convenient for some purposes, because the transformation $\dfrac{1-z}{1+z}$ not only solves the problem but additionally is equal to its inverse transformation.