

Complex Variables

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. Simplify the complex expression $\frac{(1+i)^{12}}{(1-i)^9}$ to the standard form $x + yi$, where x and y are real numbers.

Solution. Since $1 + i = \sqrt{2}e^{\pi i/4}$, and $1 - i = \sqrt{2}e^{-\pi i/4}$, the given expression can be rewritten as follows:

$$\frac{(1+i)^{12}}{(1-i)^9} = \frac{(\sqrt{2}e^{\pi i/4})^{12}}{(\sqrt{2}e^{-\pi i/4})^9} = (\sqrt{2})^{12-9} \frac{e^{12\pi i/4}}{e^{-9\pi i/4}}.$$

But $e^{12\pi i/4} = e^{3\pi i} = -1$, and $1/e^{-9\pi i/4} = e^{9\pi i/4} = e^{\pi i/4} = (1+i)/\sqrt{2}$, so the whole expression simplifies to $-(\sqrt{2})^2(1+i)$ or $-2 - 2i$.

2. If $z = x + iy$ and $f(z) = (\cos(x) + i \sin(x))e^y$, is f an analytic function? Explain.

Solution. The Cauchy–Riemann equations do not hold, so the function f is not analytic. Indeed,

$$\frac{\partial}{\partial x} (\cos(x)e^y) = -\sin(x)e^y \quad \text{and} \quad \frac{\partial}{\partial y} (\sin(x)e^y) = \sin(x)e^y,$$

so the first Cauchy–Riemann equation fails. Similarly, the second of the Cauchy–Riemann equations fails:

$$\frac{\partial}{\partial y} (\cos(x)e^y) = \cos(x)e^y \quad \text{and} \quad -\frac{\partial}{\partial x} (\sin(x)e^y) = -\cos(x)e^y.$$

Both equations are off by a minus sign, so f is actually the complex conjugate of an analytic function. In fact, $\overline{f(z)} = e^{-iz}$.

3. Evaluate the complex line integral $\int_C \frac{1}{z} dz$ when C is the line segment in the complex plane that starts at the point 1 and ends at the point i .

Solution. Since the path C is not a closed path, Cauchy's integral formula is not applicable. The two viable methods are parametrizing the path and using an antiderivative.

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Method 1. Directly parametrizing the path is feasible but leads to a tricky integral to evaluate. Simpler is to use the path-deformation principle to replace the path C with the piece of the unit circle in the first quadrant. If $z = e^{i\theta}$, then $dz = ie^{i\theta} d\theta$, so the integral can be evaluated as follows:

$$\int_0^{\pi/2} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = \int_0^{\pi/2} i d\theta = i\pi/2.$$

Method 2. The integration path lies within the region where the principal branch of the logarithm function is defined and analytic, so the integral can be evaluated as $\log(i) - \log(1)$ by using an antiderivative. The principal value of $\log(1)$ is 0, and the principal value of $\log(i)$ is $\ln|i| + i \arg(i)$, or $0 + i\pi/2$. Thus the value of the integral is $i\pi/2$.

4. Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{i^n + 2}{(3i + 4)^n} z^n$.

Solution. Apply the root test. Since $1 \leq |i^n + 2| \leq 3$ (by the triangle inequality), the squeeze theorem implies that $\lim_{n \rightarrow \infty} |i^n + 2|^{1/n} = 1$. Then

$$\lim_{n \rightarrow \infty} \left| \frac{i^n + 2}{(3i + 4)^n} z^n \right|^{1/n} = \left| \frac{z}{3i + 4} \right| \lim_{n \rightarrow \infty} |i^n + 2|^{1/n} = \frac{|z|}{5}.$$

By the root test, the series converges when $|z|/5 < 1$ and diverges when $|z|/5 > 1$. In other words, the radius of convergence of the power series is equal to 5.

5. Let C denote the circle of radius 1 centered at 0, equipped with the usual counterclockwise orientation. Evaluate

$$\int_C \frac{\cos(z) + e^z}{z^{407}} dz.$$

Solution. Method 1. Apply Cauchy's integral formula for derivatives:

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz.$$

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In this problem, $n = 406$ and $f(z) = \cos(z) + e^z$. Then

$$f^{(406)}(z) = -\cos(z) + e^z, \quad \text{so} \quad f^{(406)}(0) = -\cos(0) + e^0 = 0.$$

Accordingly, the original integral equals 0 times $2\pi i/406!$, or simply 0.

Method 2. If the integrand is expanded in a Laurent series, then the only term that contributes to the integral is the $1/z$ term. That term comes from the z^{406} term in the Maclaurin series of $\cos(z) + e^z$. In the series for $\cos(z)$, the z^{406} term has coefficient $-1/406!$, and in the series for e^z , the z^{406} term has coefficient $+1/406!$; these coefficients cancel, so the Laurent series of the integrand has no $1/z$ term. Therefore the integral is equal to 0.

6. Give an example of a Laurent series (in powers of z and $1/z$) that converges in the annulus where $3 < |z| < 4$ and in no larger annulus.

Solution. Perhaps the simplest example is

$$\sum_{n=1}^{\infty} \left(\frac{z}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{z}\right)^n.$$

The first geometric series converges precisely when $|z/4| < 1$, or $|z| < 4$; and the second geometric series converges precisely when $|3/z| < 1$, or $3 < |z|$. Therefore the whole Laurent series converges when $3 < |z| < 4$ and in no larger region.

7. Describe the image of the unit square in the complex plane (the set of all z for which $0 \leq \operatorname{Re}(z) \leq 1$ and $0 \leq \operatorname{Im}(z) \leq 1$) under the mapping that sends z to z^2 .

Solution. The bottom edge of the square maps back to the segment of the real axis between 0 and 1. The left-hand edge of the square maps to the segment of the real axis between -1 and 0 (since the square of a purely imaginary number is a negative real number). The vertex $1 + i$ maps to the point $2i$ on the imaginary axis. What about the rest of the picture?

If $z = x + iy$, then $z^2 = (x^2 - y^2) + 2xyi$. If $z^2 = w = u + iv$, then $u = x^2 - y^2$ and $v = 2xy$. Eliminating y shows that

$$u = x^2 - \frac{v^2}{4x^2}.$$

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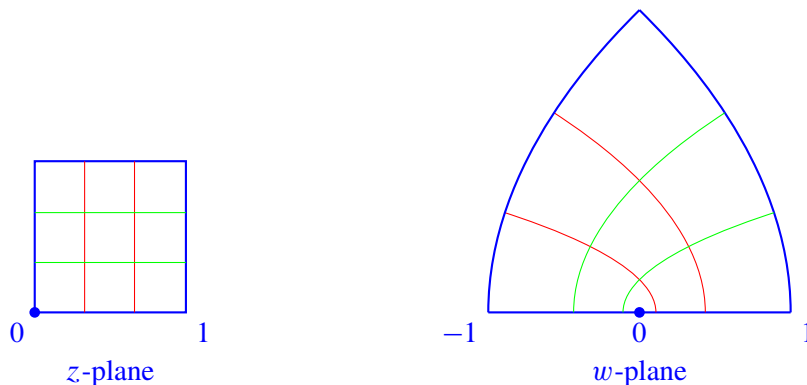
Therefore a vertical line segment inside the square in the z -plane (on which x is constant and $0 < x < 1$) maps to a piece of a parabola opening to the left in the w -plane, starting at the vertex on the positive part of the horizontal axis (corresponding to $0 = y = v$) and ending at a point in the second quadrant (corresponding to $y = 1$ and $v = 2x$ and $u = x^2 - 1 < 0$). The right-hand edge of the square (where $x = 1$) maps to a piece of the parabola on which $u = 1 - \frac{1}{4}v^2$, starting at the point 1 on the real axis and ending at the point $2i$.

Similarly, eliminating x shows that

$$u = \frac{v^2}{4y^2} - y^2.$$

Therefore a horizontal line segment inside the square in the z -plane (on which y is constant and $0 < y < 1$) maps to a piece of a parabola opening to the right in the w -plane, starting at the vertex on the negative part of the horizontal axis (corresponding to $0 = x = v$) and ending at a point in the first quadrant (corresponding to $x = 1$ and $v = 2y$ and $u = 1 - y^2 > 0$). The top edge of the square maps to a piece of the parabola on which $u = \frac{1}{4}v^2 - 1$, starting at the point -1 on the real axis and ending at the point $2i$.

By conformality, the two sets of parabolas meet at right angles. Conformality fails at the origin, where the derivative of the mapping function is equal to 0; at the origin, the angle doubles. See the figure.



(This example was discussed in class on September 15.)

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Extra credit

Use the residue theorem to show that $\int_0^\infty \frac{x^2}{1+x^5} dx = \frac{\pi/5}{\sin(2\pi/5)}$.

[This problem appeared on the qualifying examination for PhD students in the Department of Mathematics in August.]

Solution. Integrate $\frac{z^2}{1+z^5}$ over the closed curve consisting of a portion of the real axis from 0 to R (a large positive radius), an arc of the circle of radius R from angle 0 to angle $2\pi/5$, and a line segment back to the origin at angle $2\pi/5$. The integral over this closed curve equals $2\pi i$ times the residue of the integrand at the singular point $e^{\pi i/5}$: namely,

$$2\pi i \frac{z^2}{5z^4} \Big|_{z=e^{\pi i/5}}, \quad \text{or} \quad \frac{2\pi i}{5e^{2\pi i/5}}.$$

On the other hand, parametrizing the three parts of the path shows that the integral equals

$$\int_0^R \frac{x^2}{1+x^5} dx + \int_0^{2\pi/5} \frac{R^2 e^{2i\theta}}{1+R^5 e^{5i\theta}} R i e^{i\theta} d\theta + \int_R^0 \frac{r^2 e^{4\pi i/5}}{1+r^5} e^{2\pi i/5} dr.$$

The middle integral has modulus bounded above by

$$\int_0^{2\pi/5} \frac{R^3}{R^5 - 1} d\theta$$

and hence tends to 0 when $R \rightarrow \infty$. Accordingly, passing to the limit shows that

$$\frac{2\pi i}{5e^{2\pi i/5}} = (1 - e^{6\pi i/5}) \int_0^\infty \frac{x^2}{1+x^5} dx.$$

Dividing shows that

$$\begin{aligned} \int_0^\infty \frac{x^2}{1+x^5} dx &= \frac{2\pi i}{5(e^{2\pi i/5} - e^{8\pi i/5})} = \frac{2\pi i}{5(e^{2\pi i/5} - e^{-2\pi i/5})} \\ &= \frac{\pi/5}{\sin(2\pi/5)}. \end{aligned}$$