Examination 2

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Suppose $f(z) = \left(\overline{z}\right)^2$ for every z. Show that the complex derivative $f'(0)$ exists and equals 0. (Recall that the notation $\overline{z}$ means the complex conjugate of z.)

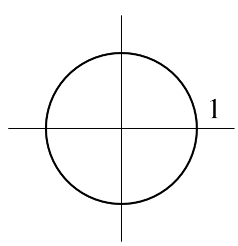
2. Determine values of the real numbers a, b, and c to make the function
$$x^2 + ay^2 + y + i(bxy + cx)$$
be an analytic function of the complex variable $x + yi$.

3. If $u(x, y) = 4x^3y - 4xy^3$, is there a function $v(x, y)$ such that $u(x, y) + iv(x, y)$ is an analytic function? Explain.

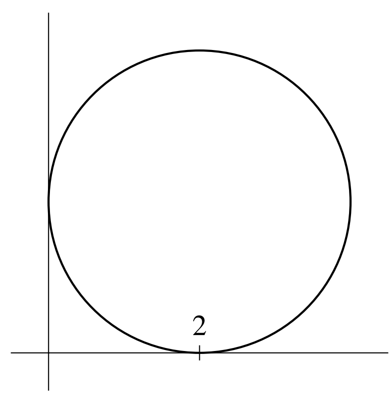
4. The complex tangent and secant functions are defined by analogy with the real counterparts: $\tan(z) = \dfrac{\sin(z)}{\cos(z)}$ and $\sec(z) = \dfrac{1}{\cos(z)}$. Is it correct to say that $\tan(z)$ is an analytic function having derivative $(\sec(z))^2$? Explain why or why not.

5. Suppose f is an analytic function defined on $\{\, z \in \mathbb{C} : \operatorname{Im}(z) > 0 \,\}$, the upper half-plane. Given the information that
$$f(f(z)) = z \qquad \text{and} \qquad f'(z) = \frac{1}{z^2} \qquad \text{for every } z,$$
find the most general possible expression for $f(z)$.

6. Determine values of the complex numbers a, b, c, and d to ensure that if $w = \dfrac{az + b}{cz + d}$, then the unit circle centered at 0 in the z-plane maps to the circle of radius 2 in the first quadrant of the w-plane tangent to the coordinate axes. See the figure.



z-plane



w-plane

Extra Credit Problem. Show that if u is the real part of a function, and v is the imaginary part, then the Cauchy–Riemann equations for u and v take the following form in polar coordinates:
$$r\,\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \text{and} \qquad r\,\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$