Final Examination

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. There is more than one value of the complex variable z for which $z^8 = z^5$. Find the solution that has the largest imaginary part.

2. Suppose $u(x, y) = 4x^3y - 4xy^3$. Find a function $v(x, y)$ such that $u + iv$ is an analytic function of the complex variable $x + iy$.

3. Suppose $w = e^z$, the complex exponential function. Determine the image in the w-plane of the set $\{ z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi \}$, a horizontal strip in the z-plane.

4. Determine the real part of the line integral $\displaystyle\int_C z^2\,dz$, where C is the piecewise linear path that moves horizontally along the real axis from 0 to 1 and then vertically from 1 to $1 + i$.

5. Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{i^n + n}{e^n + n}\right) z^n$.

6. Answer your choice of **either** part (a) **or** part (b).

   (a) Explain why there cannot be a fractional linear transformation that maps the square with vertices 0, 1, $1 + i$, and i to the rectangle with vertices 0, 2, $2 + i$, and i.

   (b) In calculus class, you once learned the method of partial fractions, according to which a function like
   $$\frac{z^2 + 1}{(z-1)(z-2)(z-3)(z-4)(z-5)}$$
   can be rewritten in the form
   $$\frac{c_1}{z-1} + \frac{c_2}{z-2} + \frac{c_3}{z-3} + \frac{c_4}{z-4} + \frac{c_5}{z-5}$$
   for certain constants $c_1, \ldots, c_5$. From the point of view of this course, the number c_j can be interpreted as the residue of the function at the point where $z = j$.

   Determine the value of the constant c_3.

Extra Credit Problem. A student argues as follows: If $z = 1/w$, then $dz = -(1/w^2)\,dw$, and $1/z = w$, so making a change of variable in the integral shows that

$$2\pi i = \int_{|z|=1} \frac{1}{z}\,dz = \int_{|w|=1} w\left(-\frac{1}{w^2}\right)\,dw = \int_{|w|=1} -\frac{1}{w}\,dw = -2\pi i.$$

But $2\pi i \neq -2\pi i$, so something went wrong in the calculation. Where is the mistake?