Final Examination

Instructions Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Prove that if z is a complex number, then $|z| = 1$ if and only if $|2z - 1| = |2 - z|$.

2. Suppose a and b are real numbers, and
$$f(x + iy) = -3x^2y + ay^3 + 2x^2 + by^2 + iv(x, y)$$
for some real-valued function $v(x, y)$. Determine the values that a and b must have in order for f to be an analytic function.

3. Give precise statements of *two* of the following four items.
   - Cauchy–Riemann equations
   - Green's theorem
   - Cauchy's theorem
   - Liouville's theorem

4. Give a concrete example of a power series $\sum_{n=1}^{\infty} a_n z^n$ for which the radius of convergence is equal to 4.

5. Suppose the rational function
$$\frac{z}{(z-1)(z-2)}$$
is expanded in a Laurent series in powers of z and z^{-1} that converges when $1 < |z| < 2$. Determine the coefficient of z^{407} in the series.

6. Evaluate the integral
$$\int_{|z-1|=2} \frac{z}{(z^2-4)\sin(z)}\, dz,$$
where the integration path is a circle with center 1 and radius 2, oriented counterclockwise (as usual).

Extra Credit

Find a conformal mapping that maps $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane) onto $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk) and takes the point 1 to the point 0.