Announcement

Exam 1 takes place Thursday, September 28.

Please bring your own paper to work on.

The material covered is Chapter I
with the following exceptions:

- Section 3 on stereographic projection
- Riemann surfaces

Reminders on logarithms and powers

$$\log(z) = \ln|z| + i \arg(z)$$
$$z^w = e^{w \log(z)}$$

(multiple possible values)

More on exponential and trigonometric functions

$$e^{iz} = \cos(z) + i\sin(z)$$
$$e^{-iz} = \cos(z) - i\sin(z)$$

Averaging the two formulas shows that $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$.

Similarly, subtracting shows that $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$.

The hyperbolic functions are twisted versions:

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad \text{and} \qquad \sinh(z) = \frac{e^{z} - e^{-z}}{2}.$$

Exercises

1. Find all values of the complex variable z for which $\cos(z) = \sin(z)$.

2. Find all values of the complex variable z for which $\cosh(z) = \sinh(z)$.

3. You know that $\cos^2(z) + \sin^2(z) = 1$.
   What is the relationship between $\cosh^2(z)$ and $\sinh^2(z)$?

Solution to Exercise 1

Rewrite the equation in terms of the complex exponential function:

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{or} \qquad ie^{iz} + ie^{-iz} = e^{iz} - e^{-iz}.$$

Collect common terms to get $(1+i)e^{-iz} = (1-i)e^{iz}$.

Multiply by e^{iz} and divide by $1-i$ to get $\dfrac{1+i}{1-i} = e^{2iz}$.

Now $\dfrac{1+i}{1-i}$ simplifies to i, so

$$2iz = \log(i) = \ln|i| + i\arg(i) = 0 + i\left(\frac{\pi}{2} + 2n\pi\right).$$

Divide by $2i$ to get the final answer,

$$z = \frac{\pi}{4} + n\pi, \qquad \text{where } n \text{ is an arbitrary integer.}$$

Solution to Exercise 2

Rewrite the equation in terms of the exponential function:

$$\frac{e^z + e^{-z}}{2} = \frac{e^z - e^{-z}}{2} \qquad \text{or} \qquad e^z + e^{-z} = e^z - e^{-z}.$$

Subtract e^z from both sides to get $e^{-z} = -e^{-z}$. Now multiply both sides by e^z to get $1 = -1$, a contradiction.

The conclusion is that the original equation has no solution!

Solution to Exercise 3

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \text{so} \qquad \cosh^2(z) = \frac{1}{4}\left(e^{2z} + 2 + e^{-2z}\right).$$

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \qquad \text{so} \qquad \sinh^2(z) = \frac{1}{4}\left(e^{2z} - 2 + e^{-2z}\right).$$

Subtracting the second equation from the first shows that

$$\cosh^2(z) - \sinh^2(z) = 1.$$

Remark. The reason for the name "hyperbolic functions" is that the equation $x^2 - y^2 = 1$ represents a hyperbola in the x–y plane.

What should $\cos^{-1}(z)$ mean?

$w = \cos^{-1}(z) \iff \cos(w) = z$
Can we solve for w in terms of z?

Write $\quad \dfrac{e^{iw} + e^{-iw}}{2} = z \quad$ or $\quad e^{iw} - 2z + e^{-iw} = 0.$
Multiply by e^{iw} to get $\left(e^{iw}\right)^2 - 2z\left(e^{iw}\right) + 1 = 0.$
By the quadratic formula,

$$e^{iw} = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1}.$$

Therefore $iw = \log\left(z \pm \sqrt{z^2 - 1}\right)$, so
$\cos^{-1}(z) = -i \log\left(z \pm \sqrt{z^2 - 1}\right)$ (infinitely many values).

Assignment (not to hand in)

Make yourself flash cards for all the main concepts and formulas from Chapter I.