# Exam results

- Mean 75
- Median 76
- Maximum 104

Grade computation: $10 + \sum_{j=1}^{6} n_j + C$, where $0 \leq n_j \leq 15$, and $0 \leq C \leq 10$.

Solutions are posted.

# Reminders on the derivative from real calculus

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

# The complex derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Is it the same story as in real calculus?
Not quite, because $h \to 0$ as a two-dimensional limit.

**Example**

$f(z) = z$:

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{z+h-z}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

**Example**

If $f(z) = \overline{z}$, then $f'(z) = \lim_{h \to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}$.
If $h \to 0$ through real values, the limit is 1, but if $h \to 0$ through purely imaginary values, the limit is $-1$. So the two-dimensional limit does not exist: $f'(z)$ does not exist.

# A necessary condition for complex differentiability

Compute $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ in two ways.

When $h \to 0$ through real values, the limit equals $\frac{\partial f}{\partial x}$.

But if $h \to 0$ through purely imaginary values, the limit equals $\frac{1}{i} \cdot \frac{\partial f}{\partial y}$.

For $f$ to be differentiable in the complex sense, these two limits must match, so a necessary requirement is that

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

# Examples

**Example**

Suppose $f(z) = |z|^2$.
$\frac{\partial f}{\partial x} = 2x$, and $-i\frac{\partial f}{\partial y} = -2iy$. Conclusion: $f$ does not have a complex derivative, except at the one point where $x = 0 = y$ (that is, $z = 0$). Moreover, $f'(0) = 0$.

**Example**

Suppose $f(x + iy) = x^2 - yi$.
A similar calculation shows that no complex derivative exists unless $x = -1/2$.

# The Cauchy–Riemann equations

Complex form:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Real form for $f = u + iv$:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

# Assignment

1. Suppose $f(x + iy) = e^x - iy$. Is this function complex differentiable?
2. Suppose $f(z) = |z|^3$. Is this function complex differentiable?