Recap: The Cauchy–Riemann equations

Complex form:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Real form for $f = u + iv$:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

If f is to have a complex derivative, then these equations must be satisfied.

The founders

Augustin-Louis Cauchy
(1789–1857)

Bernhard Riemann
(1826–1866)

Rules for derivatives

The power rule, product rule, quotient rule, and chain rule carry over from real calculus to complex calculus.

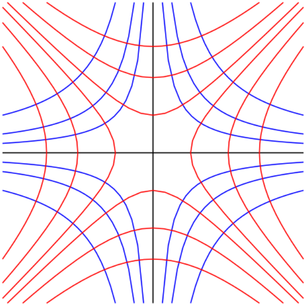
Example

If $f(z) = iz^2 + e^{iz}\sin(\pi z)$, then
$f'(z) = 2iz + ie^{iz}\sin(\pi z) + e^{iz}\pi\cos(\pi z)$

Where something new happens is the case of functions that involve $\overline{z}$ or $|z|$, or functions that are presented in terms of x and y instead of z.

Example of orthogonal curvilinear coordinate system

Suppose $w = f(z) = z^2 = (x + yi)^2 = x^2 - y^2 + 2ixy$.
If $w = u + iv$, then $u(x, y) = x^2 - y^2$, and $v(x, y) = 2xy$.
In real calculus, the curves where $u(x, y) = \text{constant}$
are called *level curves* of u.
The level curves of u and of v are families of mutually orthogonal hyperbolas:



Reminders on the gradient vector from real calculus

If $u(x, y)$ is a real function, and c is a constant, then the equation $u(x, y) = c$ defines a level curve of u.

The direction perpendicular (or orthogonal or normal) to the level curve is given by the gradient vector:

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right).$$

Example

Suppose $u(x, y) = x^2 + y^2$ and $c = 1$.
Then $\nabla u = (2x, 2y)$.
This vector is orthogonal to the circle defined by the equation $x^2 + y^2 = 1$.

Cauchy–Riemann equations imply orthogonal gradients

If u and v satisfy the Cauchy–Riemann equations, then the two gradient vectors have dot product
(or inner product or scalar product)
as follows:

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

Deduction: The level curves of u and the level curves of v are orthogonal to each other.
(In real calculus, such families of curves are known as *orthogonal trajectories*.)

Assignment

- Section II.2, Exercise 1(c)
- Section II.2, Exercise 3
- Section II.3, Exercise 2