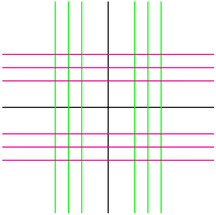
Recap: The Cauchy–Riemann equations

Real form for $f = u + iv$:

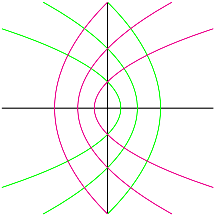
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

If f is to have a complex derivative, then these equations must be satisfied.

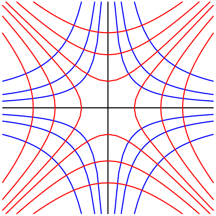
Another view of level curves: $w = z^2$



z plane



w plane



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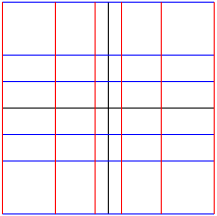

w plane

Exercise: Cauchy–Riemann and orthogonal trajectories

Suppose $u(x, y) = x^3 - 3xy^2 + y$.

Find a function $v(x, y)$ such that $u + iv$ will be a complex differentiable function.

Answer: $v(x, y) = 3x^2y - y^3 - x$ (plus an optional constant).
Moreover, $f(z) = z^3 - iz$ (plus an optional imaginary constant).

Some terminology

A subset S of $\mathbb{C}$ is called *open* if the set contains a whole disk around each of its points.

If a function f is complex differentiable at every point of an open set S, then f is called *analytic* on the set S.

A *domain* is an open set that is connected (all one piece).

We usually suppose that the domain of a function (the set where the function is defined) is a domain (a connected open set).

Harmonic functions

A function $u(x, y)$ is called *harmonic* if (u has continuous second-order partial derivatives) and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Two standard abbreviations for this equation (Laplace's equation) are $\nabla^2 u = 0$ and $\Delta u = 0$.

Analytic functions and harmonic functions

Theorem

If $f = u + iv$, and f is analytic, then u and v are harmonic.

Proof.

Differentiate the Cauchy–Riemann equations.

□

Harmonic conjugates

If $u(x, y)$ is a harmonic function, then a *harmonic conjugate of* u is a harmonic function $v(x, y)$ such that $u + iv$ is an analytic function.

Assignment

- Section II.3, Exercise 3
- Section II.5, Exercise 2