Announcement

No office hour on Monday, November 6.

(I will be on an airplane.)

Reminders from last time

Theorem (Green's theorem)

If C is a simple closed curve, oriented counterclockwise, bounding a region G, and if the functions $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on $G \cup C$, then

$$\int_C (P\,dx + Q\,dy) = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Theorem (Cauchy's integral theorem)

If $f(z)$ is analytic on and inside a simple closed curve C, then

$$\oint_C f(z)\,dz = 0.$$

Terminology for differentials

A differential $P(x, y)\,dx + Q(x, y)\,dy$ is called *closed* when

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Green's theorem implies that the integral of a closed differential around the boundary of a domain always equals 0.

A differential $P(x, y)\,dx + Q(x, y)\,dy$ is called *exact* when there exists a function h for which

$$P(x, y)\,dx + Q(x, y)\,dy = dh := \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy.$$

Example

The differential $2x\,dx + 3y^2\,dy$ is both closed and exact.
$(h = x^2 + y^3)$

Analytic functions and differentials

If $f(z)$ is analytic, then $f(z)\,dz$ always is a closed differential.
Namely $f(z)\,dz = f(z)\,dx + f(z)\,i\,dy$,
so the condition for being closed says that $\dfrac{\partial f}{\partial x}\,i = \dfrac{\partial f}{\partial y}$,
which is equivalent to the Cauchy–Riemann equations.

The differential $f(z)\,dz$ is exact precisely when f has a complex antiderivative (also called a *primitive*).
Indeed, if h is complex differentiable and $h' = f$, then

$$dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy = f(z)\,dx + if(z)\,dy = f(z)\,dz.$$

Harmonic functions and differentials

If $u(x, y)$ is harmonic, then $-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ always is a closed differential. Indeed, the condition for being closed says that

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2},$$

which is equivalent to Laplace's equation.

The differential $-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ is exact precisely when u has a harmonic conjugate function. Indeed, $dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$, which matches the given differential precisely when the Cauchy–Riemann equations hold.

Assignment

- Exercise 2 in Section III.1.
- Exercise 5 in Section III.1.
- Exercise 2(a) in Section IV.1.
  (A way to parametrize the unit circle: $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.
  Notice that you cannot divide by $m + 1$ when $m = -1$.)