Recap

Theorem (Cauchy's integral theorem)

If $f(z)$ is analytic on and inside a simple closed curve C, then

$$\oint_C f(z)\,dz = 0.$$

Cauchy's integral formula

Suppose f is analytic on and inside some simple closed curve C, and b is some point inside C.

Then $\dfrac{f(z)}{z-b}$ is *not* analytic inside C at the point b, so Cauchy's integral theorem says nothing obvious about

$$\int_C \frac{f(z)}{z-b}\,dz.$$

But the integral can be rewritten as

$$\int_C \frac{f(z)-f(b)}{z-b}\,dz + \int_C \frac{f(b)}{z-b}\,dz.$$

The first integral *is* equal to 0. The second integral, after a change of variable and a deformation of the path, equals

$$f(b)\int_{\text{circle with center } 0} \frac{1}{z}\,dz \qquad \text{or} \qquad f(b)\times 2\pi i.$$

Conclusion

Theorem (Cauchy's integral formula)

If f is analytic on a domain G together with its oriented boundary C, and b is a point inside G, then

$$f(b) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-b}\, dz.$$

Thus an analytic function is completely determined in a domain by the values of the function on the boundary!

Quiz

Complete the sentences:

1. A differential $P(x, y)\,dx + Q(x, y)\,dy$ is *closed* when ...
2. A differential $P(x, y)\,dx + Q(x, y)\,dy$ is *exact* when ...

Examples of applying Cauchy's formula

- $\displaystyle\oint_{|z|=2} \frac{3z^2}{z-1}\, dz = 2\pi i \times 3z^2\Big|_{z=1} = 6\pi i$

- $\displaystyle\oint_{|z|=2} \frac{3z^2}{z-4}\, dz = 0$ because the singularity is *outside* the curve. (Application of Cauchy's integral theorem, not the integral formula.)

- $\displaystyle\oint_{|z|=2} \frac{3z^2}{(z-1)(z-4)}\, dz = -2\pi i$ by applying the formula with $f(z) = 3z^2/(z-4)$.

Cauchy's formula for derivatives

The basic formula says that

$$f(b) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-b}\, dz.$$

Differentiate under the integral sign with respect to b to see that

$$f'(b) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-b)^2}\, dz.$$

Differentiate again to see that

$$f''(b) = \frac{2}{2\pi i} \int_C \frac{f(z)}{(z-b)^3}\, dz.$$

And in general

$$f^{(n)}(b) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-b)^{n+1}}\, dz.$$

Assignment

- Parts a,b,c,e of Exercise 1 in Section IV.4.