Follow-up on Exercise V.3.1(i)

Problem: Find the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{k!\, z^k}{k^k}$.

Solution via ratio test:

$$\lim_{k\to\infty} \left| \frac{\frac{(k+1)!\, z^{k+1}}{(k+1)^{k+1}}}{\frac{k!\, z^k}{k^k}} \right| = \lim_{k\to\infty} \left| \frac{(k+1)!\, z^{k+1} k^k}{k!\, z^k (k+1)^{k+1}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(k+1)\, z\, k^k}{(k+1)^{k+1}} \right| = \lim_{k\to\infty} \left| \frac{z\, k^k}{(k+1)^k} \right| = \lim_{k\to\infty} |z| \left(\frac{k}{k+1} \right)^k$$

$$= |z| \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}} \right)^k = \frac{|z|}{e}$$

since $\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e$. So the radius of convergence equals e.

Second solution, using Stirling's formula and root test

Theorem (Special case of Stirling's formula)

$$\lim_{k\to\infty} \frac{k!}{k^k e^{-k}\sqrt{2\pi k}} = 1.$$

(The general Stirling formula concerns an extension of the factorial function to complex numbers, the Gamma function. See Sections XIV.1 and 2 if you want to know more.)

Solution to the series problem by the root test:

$$\lim_{k\to\infty} \left|\frac{k!}{k^k} z^k\right|^{1/k} = \lim_{k\to\infty} \left|\frac{k!}{k^k e^{-k}\sqrt{2\pi k}} \left(z^k e^{-k}\sqrt{2\pi k}\right)\right|^{1/k} = |z|e^{-1}.$$

So the radius of convergence equals e (again).

Exact formula for the geometric series

Write
$$S_n = 1 + z + z^2 + \cdots + z^n.$$
Then
$$zS_n = z + z^2 + \cdots + z^n + z^{n+1},$$
so subtracting shows that
$$(1 - z)S_n = 1 - z^{n+1}.$$
Therefore
$$S_n = \frac{1 - z^{n+1}}{1 - z}.$$
If $|z| < 1$, then $\lim_{n\to\infty} z^{n+1} = 0$, so
$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1 - z} \qquad \text{when } |z| < 1.$$

Cauchy's formula implies Taylor's formula

Cauchy's formula for f analytic on and inside unit circle C says:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw \qquad (z \text{ inside } C)$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w}\cdot\frac{1}{1-\frac{z}{w}}\,dw$$

[geometric series] $$= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_C \frac{f(w)}{w}\cdot\left(\frac{z}{w}\right)^n\,dw$$

[Cauchy formula for $f^{(n)}$] $$= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n.$$

Conclusion: Taylor's formula holds for analytic functions.

Beyond Taylor series

The function $\dfrac{\sin(z)}{z^4}$ cannot be expanded in a series $\sum_{n=0}^{\infty} c_n z^n$ (because the function is not analytic at 0).

Nonetheless,

$$\frac{\sin(z)}{z^4} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z^4} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \cdots .$$

A series in positive and negative powers of the variable is called a *Laurent series*.

Application: $\displaystyle\oint_{|z|=2} \frac{\sin(z)}{z^4}\,dz = -\frac{2\pi i}{3!} = -\frac{\pi i}{3}$

Assignment (not to hand in)

Determine $\int_C f(z)\,dz$ for the following curve C: