Recap: Laurent series

A series in positive and negative powers of $(z - z_0)$ is a *Laurent series* with center z_0.

Example

Expand $\dfrac{4}{z^2 - 4}$ in a Laurent series with center 2. Solution:

$$\frac{4}{z^2-4} = \frac{4}{(z-2)(z+2)} = \frac{4}{(z-2)(z-2+4)}$$

$$= \frac{1}{z-2} \cdot \frac{1}{\frac{z-2}{4}+1} = \frac{1}{z-2} \cdot \frac{1}{1-\left(\frac{-(z-2)}{4}\right)}$$

$$\overset{\text{geometric series}}{=} \frac{1}{z-2} \cdot \left[1 + \left(\frac{-(z-2)}{4}\right) + \left(\frac{-(z-2)}{4}\right)^2 + \cdots\right]$$

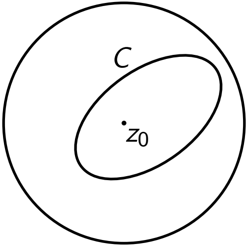
$$= \frac{1}{z-2} - \frac{1}{4} + \frac{z-2}{16} - \cdots$$

valid when $0 < |z-2| < 4$.

Residues

If $f(z)$ can be expanded in a Laurent series converging in a disk punctured at z_0, then the coefficient of $\dfrac{1}{z - z_0}$ in the series is the *residue* of f at the point z_0.

Moreover, if C is a simple closed curve in the punctured disk surrounding the point z_0, then $\oint_C f(z)\,dz = 2\pi i \times (\text{residue at } z_0)$.



Example

Since

$$\frac{4}{z^2-4} = \frac{1}{z-2} - \frac{1}{4} + \frac{z-2}{16} - \cdots ,$$

the integral

$$\oint_{|z-1|<e} \frac{4}{z^2-4}\, dz$$

equals $2\pi i \times 1$ (since the singular point 2 is inside the integration curve and the singular point -2 is outside).

But $\oint_{|z-1|<\pi} \frac{4}{z^2-4}\, dz$ is a different story, since both singular points are inside the integration curve.

Residue theorem

Theorem

When f is analytic on and inside a simple closed curve C except for some isolated singularities, then $\oint_C f(z)\,dz$ equals $2\pi i$ times the sum of the residues of f at the singular points inside C.

Computing residues: the easy case

If $f(z)$ has a first-order singularity (a *simple pole*) at z_0, then the Laurent series in a punctured disk with center z_0 has the form

$$\frac{r}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots ,$$

so r, the residue of f at z_0, equals

$$\lim_{z \to z_0} (z - z_0) f(z).$$

Example

If $f(z) = \dfrac{4}{z^2 - 4}$, then

$$\text{Res}(f, 2) = \lim_{z \to 2} (z - 2) \cdot \frac{4}{z^2 - 4} = \lim_{z \to 2} \frac{4}{z + 2} = 1.$$

Assignment for next time

Parts (a), (b), and (c) of Exercise 1 in Section VII.1.