Quiz, October 24

Suppose $w = \dfrac{z-1}{z+1}$.

1. Find the image in the w plane of the following points in the z plane: 0, 1, ∞, i.

   Solution. When $z = 0$, the corresponding value of w is $\dfrac{0-1}{0+1}$, which simplifies to -1.

   When $z = 1$, the corresponding value of w is $\dfrac{1-1}{1+1}$, which simplifies to 0.

   When $z = i$, the corresponding value of w is $\dfrac{i-1}{i+1}$, which simplifies as follows:

   $$\frac{i-1}{i+1} = \frac{i-1}{i+1} \cdot \frac{-i+1}{-i+1} = \frac{2i}{2} = i.$$

   Alternatively, observe that $i - 1 = \sqrt{2}\, e^{3\pi i/4}$, and $i + 1 = \sqrt{2}\, e^{i\pi/4}$, so

   $$\frac{i-1}{i+1} = \frac{e^{3\pi i/4}}{e^{i\pi/4}} = e^{2\pi i/4} = e^{\pi i/2} = i.$$

   To find the image of the point ∞, observe that

   $$w = \frac{z-1}{z+1} = \frac{1 - \frac{1}{z}}{1 + \frac{1}{z}}.$$

   Therefore the image of ∞ is 1.

2. Find the image in the w plane of the set $\{\, z : \operatorname{Re}(z) = 0 \,\}$ (the vertical axis in the z plane). The answer is some circle: which circle?

   Solution. There is a sneaky way to get the answer without doing elaborate calculations. Namely, three points on the image set are known from the first question: the points -1, i, and 1 are the images of the points 0, i, and ∞.

   The third point needs some explanation. Should the "ends" of the imaginary axis be $i\infty$ and $-i\infty$? No, for the following reason. Certainly $0/i = 0$, so if we declare that $1/0$ is to be designated ∞, then we are forced to accept that $i/0 = 1/(0/i) = 1/0 = \infty$. Thus there is only one point ∞, and "both ends" of every line "meet" at ∞.

   Since three points determine a circle, the problem is essentially solved by knowing that the three points -1, i, and 1 lie in the image set. But some additional information is available to pin down the solution. Linear fractional transformations are conformal mappings, so the two coordinate axes in the z plane map to image curves that cross orthogonally at the image of the point 0, which is the point -1 in the w plane. Since the coefficients of the mapping

Quiz, October 24

$\dfrac{z-1}{z+1}$ are real, the real axis in the z plane maps to the real axis in the w plane. Therefore the imaginary axis in the z plane maps to a curve in the w plane that intersects the real axis at a right angle at the point -1.

Thus the image curve in the w plane is a circle that cuts the real axis orthogonally at -1 and also passes through the points i and 1. Evidently this circle must be the unit circle centered at the origin.

To verify the solution by an explicit calculation, you could proceed as follows. The image set can be expressed as a parametric curve: namely,

$$w = \frac{iy-1}{iy+1}, \qquad y \in \mathbb{R}. \tag{*}$$

If you merely want to verify that the answer is the unit circle, then observe that

$$|w| = \left|\frac{iy-1}{iy+1}\right|.$$

To show that $|w| = 1$ is equivalent to showing that $|iy-1| = |iy+1|$. But the absolute value of a complex number is the same as the absolute value of the complex conjugate, so

$$|iy-1| = |\overline{iy-1}| = |-iy-1| = |-(iy+1)| = |iy+1|,$$

as desired.

If you are working from scratch, without a candidate for the image curve in your hands, then you could rewrite (*) as follows:

$$w = \frac{iy-1}{iy+1} = \frac{iy-1}{iy+1}\cdot\frac{-iy+1}{-iy+1} = \frac{y^2-1+2iy}{y^2+1}.$$

Express w as $u + vi$ to obtain parametric equations in the u–v plane:

$$u = \frac{y^2-1}{y^2+1} \qquad \text{and} \qquad v = \frac{2y}{y^2+1}.$$

Since the usual parametric equations for a circle involve sines and cosines, you may be surprised that these rational functions provide an alternative parametrization of a circle. To confirm that you have a circle, you would like to eliminate the parameter y to get an equation between u and v. The equation for u involves only the square of y, so you can solve to see that

$$y^2 = \frac{1+u}{1-u}.$$

You could take the square root and substitute into the equation for v, but it is simpler to avoid square roots by working instead with v^2: namely,

$$v^2 = \frac{4y^2}{(y^2+1)^2} = \frac{4\left(\dfrac{1+u}{1-u}\right)}{\left(\dfrac{1+u}{1-u}+1\right)^2} = \frac{4\left(\dfrac{1+u}{1-u}\right)}{\left(\dfrac{2}{1-u}\right)^2} = 1-u^2.$$

Quiz, October 24

And now the equation of the unit circle in the u–v plane is manifest.

3. Find the image in the w plane of $\{\, z : |z| = 1 \,\}$ (the unit circle in the z plane).

Solution. Write z in polar form: $z = e^{i\theta}$ (since $r = |z| = 1$). Then

$$w = \frac{z-1}{z+1} = \frac{e^{i\theta}-1}{e^{i\theta}+1} = \frac{e^{i\theta}-1}{e^{i\theta}+1} \cdot \frac{e^{-i\theta}+1}{e^{-i\theta}+1} = \frac{e^{i\theta}-e^{-i\theta}}{2+e^{i\theta}+e^{-i\theta}}.$$

Recall that $\cos(\theta) = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, and $\sin(\theta) = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$, so

$$w = \frac{2i\sin(\theta)}{2+2\cos(\theta)} = \frac{\sin(\theta)}{1+\cos(\theta)}i.$$

When θ runs from $-\pi$ to π, the fraction $\dfrac{\sin(\theta)}{1+\cos(\theta)}$ runs over all real numbers, positive and negative. So the unit circle in the z plane maps to the imaginary axis in the w plane.

Remark. A no-calculation solution is possible for this problem too. It is a general property of linear fractional transformations that they map circles to circles, if you interpret a line as being a special case of a circle (namely, a circle of infinite radius). Since the given transformation maps the point -1 to the point ∞, the image of the unit circle must be a circle passing through ∞, that is, a line. By conformality, this image line must intersect the real axis orthogonally at the image of the point 1, which is the point 0 in the w plane. The line perpendicular to the real axis at 0 is the imaginary axis in the w plane.