Quiz due September 7

1. If z denotes the complex number $2 - i$, then list the following real numbers in increasing order: $\operatorname{Im}(z)$, $|z|$, $\operatorname{Re}(z^2)$.

   Solution. By definition, $\operatorname{Im}(2-i) = -1$, and $|2-i| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$. Moreover, $(2-i)^2 = 4 - 4i + (-i)^2 = 4 - 4i - 1 = 3 - 4i$, so $\operatorname{Re}(z^2) = 3$. Evidently $-1 < \sqrt{5} < 3$, so the three numbers $\operatorname{Im}(z)$, $|z|$, and $\operatorname{Re}(z^2)$ were already listed in increasing order in the statement of the problem!

   Remark. This problem was supposed to be a routine exercise to confirm that you know the definitions. Notice that $\operatorname{Re}(z^2) \neq (\operatorname{Re} z)^2$: taking the real part and taking the square are operations that do not commute with each other!

2. If z denotes the complex number $1 + i$, then compute $z + \frac{1}{z}$ and express the answer in the form $a + bi$.

   Solution. By the standard trick,
   $$\frac{1}{z} = \frac{1}{1+i} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2}.$$
   Therefore
   $$z + \frac{1}{z} = 1 + i + \frac{1-i}{2} = \frac{3}{2} + \frac{1}{2}i.$$

3. Find all values of the complex variable z for which $|z-1| = |z+1|$.

   Solution. Method 1. Interpret the absolute value of a difference as a distance. The equation says that z represents a point having the same distance from the point 1 as from the point -1. Deduce from geometry that the point z lies on the perpendicular bisector of the line segment joining 1 to -1. In other words, the point z lies on the imaginary axis (that is, the y-axis).

   Method 2. Square both sides and invoke Exercise 5 from Section I.1 (which we talked about in class):
   $$|z|^2 - 2\operatorname{Re}(z) + 1 = |z|^2 + 2\operatorname{Re}(z) + 1.$$
   Move all the terms to the right-hand side to see that $0 = 4\operatorname{Re}(z)$, so $\operatorname{Re}(z) = 0$. In other words, the value of z is yi for an arbitrary real number y.

   Method 3. Replace z by $x + yi$:
   $$|x + yi - 1| = |x + yi + 1| \qquad \text{or} \qquad |(x-1) + yi| = |(x+1) + yi|.$$
   Square both sides to see that
   $$(x-1)^2 + y^2 = (x+1)^2 + y^2.$$
   Cancel the y^2 terms and expand the squares:
   $$x^2 - 2x + 1 = x^2 + 2x + 1.$$
   Move all the terms to one side to see that $x = 0$. Therefore $z = yi$, where y is arbitrary.