Review exercises

1. Determine the minimum value of $\left|\sin(\frac{\pi}{2} + iy)\right|$ when y runs over all real numbers.

2. Suppose $w = \dfrac{1+z}{1-z}$. Determine the image in the w-plane of $\{\, z : |z| < 1 \,\}$, the unit disk in the z-plane.

3. What are all the possible values of the line integral $\displaystyle\int_C \frac{1}{z^2+1}\,dz$ when C varies over all possible closed paths?

4. Can you find a rational function f with the property that the Taylor series of f with center 2 has radius of convergence equal to 3 and the Taylor series of f with center -2 has radius of convergence equal to 1?

Solution to Problem 1. Since $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$, the expression can be rewritten in terms of the complex exponential function:

$$\left|\sin\left(\frac{\pi}{2} + iy\right)\right| = \left|\frac{1}{2i}\left(e^{i\frac{\pi}{2}-y} - e^{-i\frac{\pi}{2}+y}\right)\right| = \left|\frac{1}{2i}\left(ie^{-y} - (-i)e^{y}\right)\right| = \frac{1}{2}\left(e^{y} + e^{-y}\right) = \cosh(y).$$

If you have a picture of the hyperbolic cosine function in your head, then you are done: the minimum value is 1, taken when $y = 0$.

If you do not know the graph of this function, then you can work out the answer by using tools from first-semester real calculus. Compute the derivative:

$$\cosh'(y) = \frac{1}{2}\left(e^{y} - e^{-y}\right).$$

If $y > 0$, then $e^y > 1$ and $e^{-y} < 1$, so $\cosh'(y) > 0$. By symmetry, $\cosh'(y) < 0$ when $y < 0$. Thus the function $\cosh(y)$ is decreasing when $y < 0$ and increasing when $y > 0$. Accordingly, the function has a global minimum when $y = 0$. Moreover, the value of the function at the origin is $\frac{1}{2}(e^0 + e^{-0})$, or 1. Incidentally, the graph is concave up, since $\cosh''(y) = \cosh(y) > 0$.

Solution to Problem 2. The problem can be solved by using general properties of fractional linear transformations. Since the point 1 in the z-plane maps to the point ∞ in the w-plane, the image of the unit circle (where $|z| = 1$) must be a line. Which line?

Since the coefficients of the transformation are real numbers, the real axis in the z-plane maps to the real axis in the w-plane. Fractional linear transformations are conformal, and the unit circle intersects the real axis orthogonally at the point -1 in the z-plane, so the image line intersects the real axis orthogonally at the image of -1, which is the point 0 in the w-plane.

Therefore the image of the unit circle is a vertical line in the w-plane passing through the point 0. The points inside the unit circle in the z-plane must map to a half-plane on one side of the vertical line in the w-plane. Which side?

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Since the point 0 in the z-plane maps to the point 1 in the w-plane, the image half-plane must be the right-hand half-plane, not the left-hand half-plane. Thus the image in the w-plane is the set where $\operatorname{Re}(w) > 0$.

If you prefer formulas to geometry, then you could analyze the mapping as follows:

$$w = \frac{1+z}{1-z} = \frac{1+z}{1-z} \cdot \frac{1-\overline{z}}{1-\overline{z}} = \frac{1-|z|^2+z-\overline{z}}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2} + \frac{2i\operatorname{Im}(z)}{|1-z|^2}.$$

Therefore $|z| < 1$ if and only if $\operatorname{Re}(w) > 0$. (Probably you would not think of making this calculation until you first plotted some points to get a feel for what the answer is likely to be.)

Solution to Problem 3. The integrand has singularities when $z = \pm i$, and

$$\operatorname{Res}\left(\frac{1}{z^2+1}, i\right) = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i}, \qquad \text{and} \qquad \operatorname{Res}\left(\frac{1}{z^2+1}, -i\right) = \left.\frac{1}{2z}\right|_{z=-i} = -\frac{1}{2i}.$$

If C is a simple closed curve, oriented counterclockwise, then the residue theorem implies that the value of $\int_C \frac{1}{z^2+1}\,dz$ is $2\pi i$ times the sum of the residues at the singularities inside the curve. If the curve surrounds neither singular point, then the value of the integral is 0. If the curve surrounds only the singular point i, then the value of the integral is $2\pi i \times \frac{1}{2i}$, or π. If the curve surrounds only the singular point $-i$, then the value of the integral is $-\pi$. If the curve is oriented clockwise instead of counterclockwise, then the sign of the answer changes, but the possibilities are still the three values 0, π, and $-\pi$.

Something new can happen if the curve is not simple, that is, if the curve crosses itself. A curve with self-intersections can be broken up into two or more simple closed curves, so the integral around such a curve amounts to a sum of integrals over simple closed curves. In other words, a curve that winds around one of the singular points multiple times has an integral that counts the residue at an enclosed singularity multiple times. Accordingly, it is possible to construct a curve C for which the integral equals any integer multiple of π.

There is yet another possibility. If the curve C passes *through* one of the singular points $\pm i$, where the integrand is undefined, then the integral does not have a well-defined value. (More precisely, the integral—which ultimately represents a certain limit—is divergent.)

Solution to Problem 4. The simplest series in powers of $(z-2)$ that has radius of convergence equal to 3 is the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{z-2}{3}\right)^n,$$

which sums (when $|z-2| < 3$) to the rational function

$$\frac{1}{1-\frac{z-2}{3}}, \qquad \text{or} \qquad \frac{3}{5-z}.$$

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The simplest series in powers of $(z+2)$ that has radius of convergence equal to 1 is the geometric series

$$\sum_{n=0}^{\infty}(z+2)^n,$$

which sums (when $|z+2| < 1$) to the rational function

$$\frac{1}{1-(z+2)}, \qquad \text{or} \qquad \frac{-1}{z+1}.$$

These two rational functions are not the same function, so the problem is not yet solved. One fix is to modify the first series:

$$-\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z-2}{3}\right)^n = \frac{-1/3}{1-\left(-\frac{z-2}{3}\right)} = \frac{-1}{z+1}.$$

Accordingly, the rational function $\dfrac{-1}{z+1}$ has a series expansion in powers of $(z-2)$ with radius of convergence 3 and also a series expansion in powers of $(z+2)$ with radius of convergence 1. Thus $\dfrac{-1}{z+1}$ solves the problem.

The high-level approach to this problem is to realize that the Taylor series with center z_0 for a given analytic function converges inside the largest disk centered at z_0 in which the function remains analytic (and the Taylor series converges in no larger concentric disk). So the radius of convergence of the Taylor series of a function can be read off by finding the distance from the center point to the nearest singular point of the function.

Since the only singularity of the function $\dfrac{1}{z+1}$ is the point -1, the radius of convergence of the Taylor series about the point 2 is the distance from 2 to -1: namely, radius 3. And the radius of convergence of the Taylor series about the point -2 is the distance from -2 to -1: namely, radius 1. So the function $\dfrac{1}{z+1}$ solves the problem. Many other solutions are possible: the function $\dfrac{42z+17}{(z+1)^3}$ works for the same reason.