Examination 1

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Which of the two complex numbers $(1-i)^{40}$ and $(1-i)^7$ has the larger imaginary part? Explain how you know.

   Solution. The complex number $1-i$ has angle (argument) equal to $-\pi/4$, so the power $(1-i)^{40}$ has angle $-40\pi/4$, or -10π, which is equivalent to angle 0. Therefore the complex number $(1-i)^{40}$ lies on the real axis, hence has imaginary part equal to 0.

   Similarly, the complex number $(1-i)^7$ has angle $-7\pi/4$ (equivalent to angle $+\pi/4$), which puts the complex number in the first quadrant. Numbers in the first quadrant have positive imaginary part.

   The conclusion is that $(1-i)^7$ has larger imaginary part than does $(1-i)^{40}$.

2. Determine the set of values of the complex variable z for which

$$\operatorname{Re}\left(\frac{2}{z}\right) > 1,$$

   and sketch a picture representing this set.

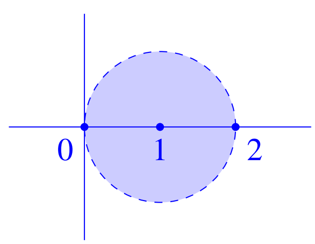
   Solution. Write z as $x+yi$ and observe that $\dfrac{2}{z} = \dfrac{2\bar{z}}{z\bar{z}} = \dfrac{2(x-yi)}{x^2+y^2}$. Therefore the inequality can be rewritten in the following form:

$$\frac{2x}{x^2+y^2} > 1, \qquad \text{or} \qquad 2x > x^2 + y^2.$$

   Subtract $2x$ from both sides and then add 1 to see that

$$1 > x^2 - 2x + 1 + y^2, \qquad \text{or} \qquad 1 > (x-1)^2 + y^2.$$

   This inequality represents a disk bounded by a circle of radius 1 with center at the point $(1,0)$, as shown in the figure.



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3. Determine the three values of the complex variable z for which $z^3 = i$. Write each value in Cartesian form as $a + bi$.

   Solution. First write i in polar form as $e^{i\pi/2}$, or equivalently as $e^{5\pi i/2}$ or $e^{-3\pi i/2}$. Three values of z for which $z^3 = i$ are therefore $e^{i\pi/6}$ and $e^{5\pi i/6}$ and $e^{-\pi i/2}$. By Euler's formula, these values can be rewritten as $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $-i$.

4. Find all values of the complex variable z for which

   $$(\log z)^2 = 1$$

   (all possible branches of the logarithm). Explain your reasoning.

   Solution. The only numbers whose square can equal 1 are 1 and -1. Thus $\log z = \pm 1$. Exponentiate to see that z must be either e^1 or e^{-1}, that is, either e or $1/e$. These are the only two possible values for z, whatever branch of the logarithm is chosen.

5. Does complex conjugation commute with taking the exponential? In other words, is it correct to say that

   $$\exp(\,\overline{z}\,) = \overline{\exp(z)}\ ?$$

   Explain why or why not.

   Solution. This formula is correct. In fact, the formula is Exercise I.5.3 in the textbook.

   For a proof, write z as $x + yi$. Then $\overline{z} = x - yi$. By Euler's formula,

   $$\exp(\,\overline{z}\,) = e^{x-yi} = e^x(\cos(-y) + i\sin(-y)) = e^x(\cos(y) - i\sin(y))$$

   (since the cosine is an even function, and the sine is an odd function). On the other hand,

   $$\exp(z) = e^x(\cos(y) + i\sin(y)), \qquad \text{so} \qquad \overline{\exp(z)} = e^x(\cos(y) - i\sin(y)).$$

   The expressions for $\exp(\,\overline{z}\,)$ and $\overline{\exp(z)}$ do indeed match.

6. Is it correct to say that $\left(2^2\right)^i = 2^{2i}$? Explain why or why not.

   Solution. The answer depends on how the question is interpreted. Are the complex powers to be viewed as principal values or as sets of all possible values?

   If you use principal values, then the answer is affirmative. Indeed,

   $$\begin{aligned} \left(2^2\right)^i &= \exp(i\,\mathrm{Log}\,2^2) = \exp\left(i(\ln 2^2 + 0i)\right) = \exp(2i\ln 2), \quad \text{and} \\ 2^{2i} &= \exp(2i\,\mathrm{Log}\,2) = \exp\left(2i(\ln 2 + 0i)\right) = \exp(2i\ln 2). \end{aligned}$$

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These answers match.

But if instead you consider the sets of all possible values, then the answer is negative. The values of the left-hand side can be expressed as follows:

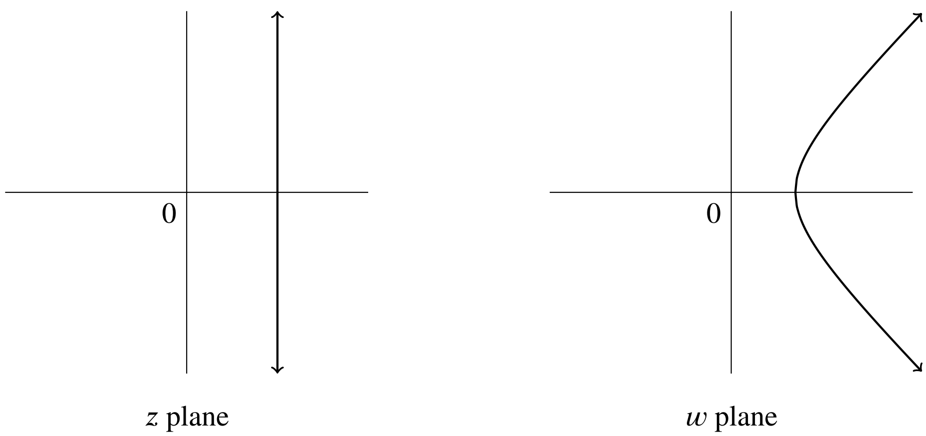
$$\exp\left(i \log 2^2\right) = \exp\left(i(\ln 2^2 + 2\pi n i)\right) = \exp\left(2i \ln 2 - 2\pi n\right),$$

where n is an arbitrary integer. The values of the right-hand side can be computed by the same method:

$$\exp\left(2i(\ln 2 + 2\pi n i)\right) = \exp\left(2i \ln 2 - 4\pi n\right),$$

where n is an arbitrary integer. Every value of the second form is included in the first list, but the first list has some values, such as $\exp(2i \ln 2 - 2\pi)$, that are not included in the second list. The sets of all possible values of the two sides are not identical.

Extra Credit Problem. If $w = f(z)$, and the curves with arrows shown in the pictures below correspond with respect to this function, could $f(z)$ be equal to z^2 or e^z or $\sin(z)$ or none of these? Explain how you know.



Solution. Points high up on the indicated line in the z plane have argument slightly less than $\pi/2$, so the image of such a point under the function z^2 has argument slightly less than π, hence lies in the second quadrant. But the picture in the w plane stays away from the second quadrant, so $f(z)$ cannot be z^2. Moreover, you previously solved Exercise I.4.1(b) and determined that a vertical line in the right-hand half of the z plane maps under z^2 to a parabola opening to the left in the w plane. You could cite this exercise as another reason that $f(z)$ cannot be z^2.

Since $e^{x+yi} = e^x e^{yi}$, and $|e^{yi}| = 1$, a vertical line in the z plane maps under the exponential function onto a circle in the w plane of radius e^x. The picture in the w plane is not a circle, so $f(z)$ cannot be e^z. See also the figure in the textbook preceding the exercises for Section I.5, as well as Exercise I.5.2(a) that you previously solved.

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Could the picture correspond to $\sin(z)$, or must $f(z)$ be some entirely different function? To understand the sine function, you could say that

$$\begin{aligned}
\sin(x + yi) &= \frac{1}{2i}\left(e^{ix-y} - e^{-ix+y}\right) \\
&= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right) \\
&= (\sin x)(\cosh y) + i(\cos x)(\sinh y).
\end{aligned}$$

Accordingly, if $u + vi = \sin(x + yi)$, then $u = (\sin x)(\cosh y)$, and $v = (\cos x)(\sinh y)$. If x is a number for which $\sin x$ is positive, then u is positive too, so $u+vi$ lies in the right-hand half-plane. If additionally $\cos x \neq 0$, then

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1.$$

The interpretation of this equation is that a vertical line in the z plane (on which y varies but x is constant) has an image in the w plane that is a branch of a hyperbola. The conclusion is that the picture is consistent with $f(z)$ being $\sin(z)$, as long as the indicated vertical line in the z plane crosses the x axis at a point where $0 < \sin x < 1$.