Complex Variables

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. State the following:

   (a) De Moivre's theorem about powers of complex numbers;

   (b) the Cauchy–Riemann equations.

   Solution. De Moivre's theorem says that

   $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

   when θ is an arbitrary angle and n is an arbitrary natural number (positive integer). We observed in class that the formula holds too when n is a negative integer. More generally,

   $$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

   The Cauchy–Riemann equations, which characterize analyticity of a function $u(x, y) + iv(x, y)$, say that

   $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

2. Which of the complex numbers $\left(\frac{1+i}{2}\right)^4$ and $\left(\frac{1}{\sqrt{3}-i}\right)^2$ has bigger imaginary part? Which of these two complex numbers has bigger modulus? Explain how you know.

   Solution. Method 1: Use the trigonometric form to compute the powers. Since $\frac{1+i}{2} = \frac{\sqrt{2}}{2}(\cos(\pi/4) + i\sin(\pi/4))$, taking the fourth power by invoking De Moivre's formula shows that

   $$\left(\frac{1+i}{2}\right)^4 = \left(\frac{\sqrt{2}}{2}\right)^4 \left(\cos(\pi) + i\sin(\pi)\right) = -\frac{1}{4}.$$

Similarly, $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2(\cos(-\pi/6) + i\sin(-\pi/6))$. The case of De Moivre's theorem with exponent equal to -1 shows that

$$\frac{1}{\sqrt{3} - i} = \frac{1}{2}\left(\cos(\pi/6) + i\sin(\pi/6)\right),$$

so

$$\left(\frac{1}{\sqrt{3} - i}\right)^2 = \frac{1}{4}\left(\cos(\pi/3) + i\sin(\pi/3)\right) = \frac{1}{4}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right).$$

Accordingly, the second of the two given complex numbers has larger imaginary part ($\sqrt{3}/8$ versus 0), but the two complex numbers have the same modulus (namely, $1/4$).

Method 2: Expand in rectangular form. Since

$$\left(\frac{1+i}{2}\right)^2 = \frac{1 + 2i + i^2}{4} = \frac{2i}{4} = \frac{i}{2},$$

squaring a second time shows that

$$\left(\frac{1+i}{2}\right)^4 = \left(\frac{i}{2}\right)^2 = -\frac{1}{4}.$$

On the other hand,

$$\left(\frac{1}{\sqrt{3} - i}\right)^2 = \frac{1}{3 - 2\sqrt{3}\,i + i^2} = \frac{1}{2 - 2\sqrt{3}\,i} = \frac{1}{2} \cdot \frac{1}{1 - \sqrt{3}\,i}$$
$$= \frac{1}{2} \cdot \frac{1}{1 - \sqrt{3}\,i} \cdot \frac{1 + \sqrt{3}\,i}{1 + \sqrt{3}\,i} = \frac{1}{2} \cdot \frac{1 + \sqrt{3}\,i}{4}.$$

The conclusion is the same as before.

3. Consider the sequence $z_1, z_2, \ldots$ of complex numbers defined recursively as follows:

$$z_1 = 3 + 2i \qquad \text{and} \qquad z_{n+1} = \frac{i}{z_n} \quad \text{when } n \geq 1.$$

Determine all the limit points of this sequence.

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Solution. Observe that $z_2 = i/z_1$, so $z_3 = i/z_2 = i/(i/z_1) = z_1$. The pattern of terms now repeats, so the sequence is $z_1, z_2, z_1, z_2, \ldots$. Although the sequence does not converge, it has two limit points (limits of subsequences): namely, z_1 and z_2.

This deduction is independent of the particular value of z_1. In the specific case that $z_1 = 3 + 2i$, the value of z_2 is $i/(3 + 2i)$ or $i(3 - 2i)/13$ or $(2 + 3i)/13$. Thus the limit points of the given sequence are $3 + 2i$ and $(2 + 3i)/13$.

4. Describe geometrically the set of points z in the complex plane satisfying the property that

$$|z - 1| = \operatorname{Im}(z).$$

Solution. If $z = x+iy$, then the equation says that $|x+iy-1| = y$. First of all, this equation implies that $y \geq 0$, since the modulus of a complex number represents a length and cannot be negative. Second, squaring the equation and invoking the definition of modulus implies that $(x - 1)^2 + y^2 = y^2$, or $(x - 1)^2 = 0$, or $x = 1$. Thus the equation represents the top half of a vertical line with abscissa equal to 1.

If your answer was the whole line, then you overlooked that squaring both sides of an equation loses information; squaring is not a reversible step unless all the quantities involved are positive numbers.

5. If $f(x + iy) = x^3 - y^3 + 3ix^2y$, is the function f analytic? Explain why or why not.

Solution. The function is not analytic in any domain in the complex plane, for the Cauchy–Riemann equations do not hold. The real part u equals $x^3 - y^3$, and the imaginary part v equals $3x^2y$. Now

$$\frac{\partial u}{\partial x} = 3x^2 = \frac{\partial v}{\partial y},$$

so the first Cauchy–Riemann equation does hold at every point. But

$$\frac{\partial u}{\partial y} = -3y^2 \qquad \text{and} \qquad -\frac{\partial v}{\partial x} = -6xy,$$

so the second Cauchy–Riemann equation holds only when $y = 0$ or $y = 2x$ (equations that represent a pair of lines). Thus there is no open set in the plane on which both Cauchy–Riemann equations are valid.

6. Evaluate the integral $\int_C (\overline{z} - z)\, dz$, where C is the straight line segment joining the point $(0, 0)$ to the point $(1, 3)$ in the complex plane.

Solution. The path C is part of the straight line on which $y = 3x$. The simplest parametrization of C is obtained by setting $x(t)$ equal to t and $y(t)$ equal to $3t$, where $0 < t < 1$. On the path, $\overline{z} - z = -2iy = -6it$, and $dz = dx + i\, dy = dt + i \cdot 3\, dt = (1 + 3i)\, dt$. Therefore the integral can be evaluated as follows:

$$\begin{aligned} \int_C (\overline{z} - z)\, dz &= \int_0^1 -6it(1 + 3i)\, dt = -6i(1 + 3i) \int_0^1 t\, dt \\ &= \frac{-6i(1 + 3i)}{2} = 9 - 3i. \end{aligned}$$

Extra credit

I typed "`real part of cube root of -8`" into WolframAlpha, and I received back the answer 1 instead of the expected value of -2. Explain what decision Wolfram's programmers must have made that resulted in the computer giving the answer 1.

Solution. Every nonzero complex number has three cube roots. Since the real numbers are a subset of the complex numbers, every nonzero real number has three complex cube roots. In the case at hand,

$$\begin{aligned} -8 &= 8(\cos(\pi) + i \sin(\pi)) \\ &= 8(\cos(3\pi) + i \sin(3\pi)) \\ &= 8(\cos(5\pi) + i \sin(5\pi)), \end{aligned}$$

so the three values of the cube root are

$$\begin{aligned} &2(\cos(\pi/3) + i \sin(\pi/3)) = 1 + \sqrt{3}\, i, \\ &2(\cos(\pi) + i \sin(\pi)) = -2, \\ \text{and } &2(\cos(5\pi/3) + i \sin(5\pi/3)) = 1 - \sqrt{3}\, i. \end{aligned}$$

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The programmers had to give the computer a general rule (an algorithm) for which root of a complex number to choose. Presumably, the programmers told the computer to choose the root that has the smallest non-negative angle, the so-called *principal value*. In the case at hand, the computer would pick the cube root with argument $\pi/3$, corresponding to the value $1 + \sqrt{3}\,i$. Taking the real part of this cube root produces the answer 1.