Final Examination

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. There is more than one value of the complex variable z for which $z^8 = z^5$. Find the solution that has the largest imaginary part.

   Solution. The natural method is to find *all* solutions and then to identify the one having largest imaginary part.

   The equation says that $z^8 - z^5 = 0$, or $z^5(z^3 - 1) = 0$. Therefore either $z = 0$ or $z^3 - 1 = 0$. Since 1 can be written as e^0 or $e^{2\pi i}$ or $e^{4\pi i}$, the values of z for which $z^3 = 1$ are $e^{0/3}$ and $e^{2\pi i/3}$ and $e^{4\pi i/3}$. By Euler's formula, these values simplify to 1 and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Accordingly, the solution to the original equation having the largest imaginary part is the complex number $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Indeed, this value is the only solution that has positive imaginary part.

2. Suppose $u(x, y) = 4x^3y - 4xy^3$. Find a function $v(x, y)$ such that $u + iv$ is an analytic function of the complex variable $x + iy$.

   Solution. First notice that $u_{xx} + u_{yy} = 24xy - 24xy = 0$, so u is a harmonic function. Therefore a function v with the indicated property does exist.

   To find v, apply the Cauchy–Riemann equations. One of the equations says that

   $$v_y = u_x = 12x^2y - 4y^3.$$

   Integrating with respect to y shows that $v(x, y) = 6x^2y^2 - y^4 + g(x)$ for some function $g(x)$. Now invoke the second Cauchy–Riemann equation ($v_x = -u_y$):

   $$12xy^2 + g'(x) = v_x = -u_y = -(4x^3 - 12xy^2) = -4x^3 + 12xy^2.$$

   Comparing the two sides of this equation reveals that $g'(x) = -4x^3$, so $g(x) = -x^4 + C$ for some constant C.

   Putting the pieces together shows that $v(x, y) = 6x^2y^2 - y^4 - x^4 + C$. If you left off the constant C, your answer is still correct, for the problem demands only one solution, not the most general solution.

3. Suppose $w = e^z$, the complex exponential function. Determine the image in the w-plane of the set $\{ z \in \mathbb{C} : 0 < \operatorname{Im}(z) < \pi \}$, a horizontal strip in the z-plane.

   Solution. Points in the indicated strip have the form $x + iy$ with x unrestricted and y subject to the constraint that $0 < y < \pi$. Since $w = e^{x+iy} = e^x e^{iy}$, this representation of w in polar form reveals that $|w|$ (which equals e^x) can be an arbitrary positive number, and the argument of w (which equals y) can be an arbitrary angle between 0 and π. Accordingly, the values of w fill out the upper half-plane.

Another way to obtain the conclusion is to observe that $w = e^x(\cos y + i \sin y)$, and $\sin y > 0$ when $0 < y < \pi$, so $\operatorname{Im}(w) > 0$. This inequality is a description of the upper half of the w-plane.

4. Determine the real part of the line integral $\int_C z^2\,dz$, where C is the piecewise linear path that moves horizontally along the real axis from 0 to 1 and then vertically from 1 to $1+i$.

Solution. Method 1. The integrand has an analytic antiderivative, so the value of the integral is

$$\frac{1}{3}z^3\Big|_0^{1+i}, \qquad \text{or} \qquad \frac{1}{3}\left((1+i)^3 - 0^3\right).$$

Now $1+i = \sqrt{2}\,e^{\pi i/4}$, so $(1+i)^3 = 2\sqrt{2}\,e^{3\pi i/4} = 2(-1+i)$, so the value of the integral simplifies to $\frac{2}{3}(-1+i)$. The real part is $-\frac{2}{3}$.

Method 2. Parametrize the two parts of the path. The integral along the segment of the real axis is $\int_0^1 x^2\,dx$, which evaluates to $\frac{1}{3}$. On the vertical part of the path, $z = 1 + iy$ and $dz = i\,dy$, so the integral on this part of the path equals

$$\int_0^1 (1+iy)^2\, i\,dy, \qquad \text{or} \qquad i\int_0^1 (1 + 2iy - y^2)\,dy.$$

The latter expression evaluates to

$$i\left(y + iy^2 - \frac{1}{3}y^3\right)\Big|_0^1, \qquad \text{or} \qquad i\left(1 + i - \frac{1}{3} - 0\right), \qquad \text{or} \qquad \frac{2}{3}i - 1.$$

The real part of the whole integral is the sum of the real parts of the two pieces: namely, $\frac{1}{3} - 1$, or $-\frac{2}{3}$.

Method 3. As a variation of the second method, observe that if $z = x + iy$, then

$$z^2\,dz = \left(x^2 - y^2 + 2ixy\right)(dx + i\,dy),$$

so the real part of the integral is

$$\int_C (x^2 - y^2)\,dx - 2xy\,dy.$$

Now parametrize the two parts of the path. On the real axis, $y = 0$, so this piece of the integral is

$$\int_0^1 x^2\,dx, \qquad \text{or} \qquad \frac{1}{3}.$$

On the vertical line segment, $x = 1$ and $dx = 0$, so this piece of the integral is

$$\int_0^1 -2y\,dy, \qquad \text{or} \qquad -1.$$

The final answer is thus $\frac{1}{3} - 1$, or $-\frac{2}{3}$.

Remark. Green's theorem does not apply to this problem, for the integration path is not a closed curve; that is, the curve does not surround a region.

5. Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{i^n + n}{e^n + n}\right) z^n$.

Solution. Method 1: Root test. The initial goal is to compute the limit

$$\lim_{n\to\infty} \left| \left(\frac{i^n + n}{e^n + n}\right) z^n \right|^{1/n}.$$

Beware! An expression of the form $(A^n + B^n)^{1/n}$ does *not* simplify to $A + B$.

Computing the limit therefore looks complicated, but there is a simple device for seeing what the answer must be. The intuition is that the fastest growing terms in the numerator and denominator are all that counts, so an equivalent problem is to compute

$$\lim_{n\to\infty} \left| \frac{n}{e^n} z^n \right|^{1/n}, \qquad \text{or} \qquad \frac{|z|}{e} \lim_{n\to\infty} n^{1/n}, \qquad \text{which equals} \qquad \frac{|z|}{e}.$$

The cut-off for convergence occurs when this limit equals 1, so the radius of convergence equals e.

This intuition can be made rigorous in various ways. For instance, observe that

$$|i^n + n|^{1/n} = \left| \frac{i^n}{n} + 1 \right|^{1/n} n^{1/n}.$$

Now $\lim_{n\to\infty} n^{1/n} = 1$ (a standard limit), and

$$1^{1/n} \le \left| \frac{i^n}{n} + 1 \right|^{1/n} \le 2^{1/n}.$$

Since $\lim_{n\to\infty} 2^{1/n} = 1$ (another standard limit), the squeeze theorem implies that

$$\lim_{n\to\infty} \left| \frac{i^n}{n} + 1 \right|^{1/n} = 1.$$

Accordingly, $\lim_{n\to\infty} |i^n + n|^{1/n} = 1$. Similarly, observe that

$$(e^n + n)^{1/n} = \left(1 + \frac{n}{e^n}\right)^{1/n} (e^n)^{1/n}.$$

Since $n < e^n$, the squeeze theorem again applies to show that

$$\lim_{n\to\infty}\left(1+\frac{n}{e^n}\right)^{1/n} = 1,$$

and certainly $\lim_{n\to\infty}(e^n)^{1/n} = \lim_{n\to\infty} e = e$.

Putting all the pieces together reveals that

$$\lim_{n\to\infty}\left|\left(\frac{i^n+n}{e^n+n}\right)z^n\right|^{1/n} = \frac{\lim_{n\to\infty}|i^n+n|^{1/n}}{\lim_{n\to\infty}(e^n+n)^{1/n}}|z| = \frac{|z|}{e}.$$

This argument confirms that the radius of convergence of the original series is indeed equal to e.

Method 2: Ratio test. The goal is to compute

$$\lim_{n\to\infty}\left|\frac{\left(\frac{i^{n+1}+n+1}{e^{n+1}+n+1}\right)z^{n+1}}{\left(\frac{i^n+n}{e^n+n}\right)z^n}\right|, \qquad \text{or} \qquad \lim_{n\to\infty}\left|\frac{i^{n+1}+n+1}{i^n+n}\right|\cdot\frac{e^n+n}{e^{n+1}+n+1}\cdot|z|.$$

The limit of a product is the product of the limits, so the problem can be split into pieces. First observe that

$$\lim_{n\to\infty}\left|\frac{i^{n+1}+n+1}{i^n+n}\right| = \lim_{n\to\infty}\left|\frac{\frac{i^{n+1}+1}{n}+1}{\frac{i^n}{n}+1}\right| = 1.$$

Moreover,

$$\lim_{n\to\infty}\frac{e^n+n}{e^{n+1}+n+1} = \lim_{n\to\infty}\frac{1+\frac{n}{e^n}}{e+\frac{n+1}{e^n}} = \frac{1}{e},$$

since $\lim_{n\to\infty}\frac{n}{e^n} = 0$ (by l'Hôpital's rule, for instance) and similarly $\lim_{n\to\infty}\frac{n+1}{e^n} = 0$. Therefore the original limit equals $\frac{|z|}{e}$, so the series has radius of convergence equal to e.

6. Answer your choice of **either** part (a) **or** part (b).

   (a) Explain why there cannot be a fractional linear transformation that maps the square with vertices 0, 1, $1+i$, and i to the rectangle with vertices 0, 2, $2+i$, and i.

   Solution. Method 1. Suppose that there were such a fractional linear transformation. The sides of the square must map to the sides of the rectangle, so the four lines that contain the sides of the square must map to the four lines that contain the sides of the rectangle. These four lines in the domain all pass through the point at infinity, so the image of the point at infinity must lie on all four lines in the image. But these four lines in the image have only the point at infinity in common. Consequently, the transformation must map the point at infinity to itself.

A fractional linear transformation $\dfrac{az+b}{cz+d}$ maps ∞ to $\dfrac{a}{c}$, so if the point at infinity maps to itself, then c must be equal to 0. Consequently, the transformation reduces to the composition of a translation, a rotation, and a dilation. Each of these three types of transformation evidently maps squares to squares. So the image of a square cannot be a non-square rectangle.

Method 2. Since fractional linear transformations are conformal mappings, the corners of the square (where the sides meet at right angles) must map to the corners of the rectangle. If the corners correspond in the given order, then you can set up a system of equations satisfied by the coefficients in the fractional linear mapping $\dfrac{az+b}{cz+d}$: namely,

$$\begin{aligned} 0 &= \frac{b}{d} \\ 2 &= \frac{a+b}{c+d} \\ 2+i &= \frac{a(1+i)+b}{c(1+i)+d} \\ i &= \frac{ai+b}{ci+d}. \end{aligned}$$

This system of equations can be shown to be inconsistent, as follows.

The first equation implies that $b = 0$. Then the second equation implies that $a = 2(c+d)$. Substituting into the last equation implies that $ci+d = 2(c+d)$, or $(i-2)c = d$. Hence $a = 2(c+d) = 2(i-1)c$. Finally, substituting into the third equation shows that

$$2+i = \frac{2(i-1)(1+i)c}{c(1+i)+(i-2)c} = \frac{-4}{2i-1}, \qquad \text{a contradiction,}$$

since the absolute value of the left-hand side is $\sqrt{5}$, but the absolute value of the right-hand side is $\dfrac{4}{\sqrt{5}}$.

The solution is not yet complete, for the four corners of the square might map to the corners of the rectangle in some other order. But precomposing the transformation with a suitable rotation and translation shows that there is no loss of generality in supposing that 0 maps to 0. Since the sides of the square must map to the sides of the rectangle, the opposite vertex $1+i$ in the square must map to $2+i$. Conformal maps preserve the orientation of the boundary, so the remaining vertices 1 and i must map to 2 and i in that order. Accordingly, the calculation above does suffice to solve the problem.

(b) In calculus class, you once learned the method of partial fractions, according to which a function like

$$\frac{z^2+1}{(z-1)(z-2)(z-3)(z-4)(z-5)}$$

can be rewritten in the form

$$\frac{c_1}{z-1} + \frac{c_2}{z-2} + \frac{c_3}{z-3} + \frac{c_4}{z-4} + \frac{c_5}{z-5}$$

for certain constants $c_1, \ldots, c_5$. From the point of view of this course, the number c_j can be interpreted as the residue of the function at the point where $z = j$.

Determine the value of the constant c_3.

Solution. Method 1. View the function as $\dfrac{g(z)}{z-3}$, where

$$g(z) = \frac{z^2+1}{(z-1)(z-2)(z-4)(z-5)}.$$

The residue of the function is $g(3)$, or

$$\frac{3^2+1}{(3-1)(3-2)(3-4)(3-5)}, \qquad \text{or} \qquad \frac{10}{4}, \qquad \text{or} \qquad \frac{5}{2}.$$

Method 2. Use the rule that the residue of $\dfrac{g(z)}{h(z)}$ at a simple zero z_0 of the denominator equals $\dfrac{g(z_0)}{h'(z_0)}$. The required value is therefore

$$\left.\frac{z^2+1}{\frac{d}{dz}\left[(z-1)(z-2)(z-3)(z-4)(z-5)\right]}\right|_{z=3}.$$

Apply the product rule to the factors $(z-3)$ and $[(z-1)(z-2)(z-4)(z-5)]$ to obtain

$$\left.\frac{z^2+1}{(z-3)\frac{d}{dz}\left[(z-1)(z-2)(z-4)(z-5)\right] + \left[(z-1)(z-2)(z-4)(z-5)\right]\frac{d}{dz}(z-3)}\right|_{z=3}.$$

The first part of the denominator becomes 0 after evaluation when $z = 3$, so the whole expression reduces to

$$\left.\frac{z^2+1}{(z-1)(z-2)(z-4)(z-5)}\right|_{z=3}, \qquad \text{or} \qquad \frac{5}{2},$$

just as in the first method.

Method 3. Go back to the derivation of the rules for computing residues: multiply by $(z-3)$ to see that

$$\frac{z^2+1}{(z-1)(z-2)(z-4)(z-5)} = c_3 + (z-3)\left[\frac{c_1}{z-1} + \frac{c_2}{z-2} + \frac{c_4}{z-4} + \frac{c_5}{z-5}\right].$$

Now set z equal to 3 on both sides to deduce that $\frac{5}{2} = c_3$.

Final Examination

Extra Credit Problem. A student argues as follows: If $z = 1/w$, then $dz = -(1/w^2)\,dw$, and $1/z = w$, so making a change of variable in the integral shows that

$$2\pi i = \int_{|z|=1} \frac{1}{z}\,dz = \int_{|w|=1} w\left(-\frac{1}{w^2}\right)\,dw = \int_{|w|=1} -\frac{1}{w}\,dw = -2\pi i.$$

But $2\pi i \neq -2\pi i$, so something went wrong in the calculation. Where is the mistake?

Solution. That the integral $\int_{|z|=1} \frac{1}{z}\,dz$ evaluates to $2\pi i$ depends on the circle being oriented in the usual counterclockwise direction. But inversion interchanges the inside of the circle with the outside of the circle, hence reverses the direction of the curve. Therefore the unit circle in the w-plane is oriented backward, so a minus sign needs to be attached to $\int_{|w|=1}$. The missing minus sign is the error in the calculation.

To see the error more explicitly, consider parametrizing the initial circle by setting z equal to $e^{i\theta}$. When θ increases from 0 to 2π, the point z moves around the unit circle in the counterclockwise direction. But $w = 1/z = 1/e^{i\theta} = e^{-i\theta}$. The minus sign in the exponent means that when θ increases, the point w moves around the circle in the clockwise direction.