

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. There are six values of the complex number z for which $z^6 = -1$. Find the solution that has the largest imaginary part.

Solution. Method 1 Since $-1 = e^{\pi i + 2n\pi i}$ for every integer n , the possible values for z have the form $e^{(2n+1)\pi i/6}$, where n can be any integer. Here are these values rewritten in standard $a + bi$ form:

$$\begin{aligned} e^{\pi i/6} &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ e^{3\pi i/6} &= i \\ e^{5\pi i/6} &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \\ e^{7\pi i/6} &= -\frac{\sqrt{3}}{2} - \frac{1}{2}i \\ e^{9\pi i/6} &= -i \\ e^{11\pi i/6} &= \frac{\sqrt{3}}{2} - \frac{1}{2}i. \end{aligned}$$

Among these complex numbers, the number i is the one that has the largest imaginary part.

Method 2 Every solution is a complex number of modulus equal to 1, that is, a point on the unit circle. No point on the unit circle has imaginary part greater than 1. By inspection, however, the number i has the property that $i^6 = -1$, so i is the solution with maximal imaginary part.

2. Suppose $v(x, y) = x^3 - 3xy^2$. Find a function $u(x, y)$ such that $u(x, y) + iv(x, y)$ is an analytic function.

Solution. Apply the Cauchy–Riemann equations. Since

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -6xy,$$

integrating with respect to x shows that there is some function $g(y)$ such that

$$u(x, y) = -3x^2y + g(y).$$

Differentiating with respect to y shows that

$$-3x^2 + g'(y) = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -3x^2 + 3y^2.$$

Accordingly, $g'(y) = 3y^2$, so $g(y) = y^3 + C$ for an arbitrary real number C . Thus $u(x, y) = -3x^2y + y^3 + C$.

Remark In fact, $u(x, y) = \operatorname{Re}(iz^3 + C)$.

3. Find two distinct points z_1 and z_2 in the complex plane such that $\int_C z^2 dz = 0$, where C is the line segment joining z_1 to z_2 . (Notice that the path C is *not* a closed curve!)

Solution. The integral can be evaluated using an antiderivative as

$$\frac{1}{3}z^3 \Big|_{z_1}^{z_2}, \quad \text{or} \quad \frac{1}{3}(z_2^3 - z_1^3).$$

Accordingly, what is required is to find two distinct complex numbers z_1 and z_2 whose cubes are equal. One example is obtained by taking z_1 equal to 1 and z_2 equal to $e^{2\pi i/3}$. More generally, if z_1 is an arbitrary nonzero complex number, set z_2 equal either to $z_1 e^{2\pi i/3}$ or to $z_1 e^{4\pi i/3}$.

4. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{n + i \cos(n)}{in + 3^n}$$

converge? Explain why or why not.

Solution. This infinite series does converge. Observe that $|n + i \cos(n)| < n + 1$ by the triangle inequality, and $|in + 3^n| > 3^n$ since the modulus of a complex number is at least as big as the real part of that complex number. Therefore

$$\left| \frac{n + i \cos(n)}{in + 3^n} \right| < \frac{n + 1}{3^n}.$$

To prove convergence of the original series by the comparison test, it suffices to show that $\sum_{n=1}^{\infty} (n + 1)/3^n$ is a convergent series. Apply the ratio test:

$$\lim_{n \rightarrow \infty} \frac{(n + 1) + 1}{3^{n+1}} \Big/ \frac{n + 1}{3^n} = \frac{1}{3} \lim_{n \rightarrow \infty} \frac{n + 2}{n + 1} = \frac{1}{3}.$$

Since $1/3 < 1$, the ratio test implies that the comparison series converges, and therefore so does the original series. (Applying the comparison test together with the root test would work as well.)

5. If C denotes the unit circle centered at the origin, then which of the two integrals

$$\int_C \frac{\sin(4z)}{\cos(4z)} dz \quad \text{and} \quad \int_C \frac{\cos(4z)}{\sin(4z)} dz$$

has larger modulus? Explain how you know.

Solution. In view of the residue theorem, an equivalent problem is to decide for which case the sum of the residues at the poles of the integrand inside the curve C has larger modulus. Both integrands have only simple poles. If z_0 is a pole of $\sin(4z)/\cos(4z)$, then the residue at z_0 equals

$$\left. \frac{\sin(4z)}{-4 \sin(4z)} \right|_{z=z_0}, \quad \text{or} \quad -\frac{1}{4}.$$

Similarly, if z_1 is a pole of $\cos(4z)/\sin(4z)$, then the residue at z_1 equals

$$\left. \frac{\cos(4z)}{4 \cos(4z)} \right|_{z=z_1}, \quad \text{or} \quad \frac{1}{4}.$$

Accordingly, the question reduces to deciding which integrand has more poles inside the curve C .

The fraction $\sin(4z)/\cos(4z)$ has a pole when $4z$ is an odd multiple of $\pi/2$, or when $z = \pm\pi/8, \pm3\pi/8, \dots$. Since $3\pi > 8$, there are precisely two points inside the unit circle at which there are poles: namely, $\pm\pi/8$. On the other hand, the fraction $\cos(4z)/\sin(4z)$ has a pole when $4z$ is a multiple of π , or when $z = 0, \pm\pi/4, \pm\pi/2, \dots$. Thus the second integrand has poles at three points inside C : namely, 0 and $\pm\pi/4$.

In conclusion, the second integral has larger modulus. In fact, the second integral equals $2\pi i \times 3/4$, while the first integral equals $2\pi i \times (-1/2)$.

6. Determine the largest open annulus in which the Laurent series

$$\cdots + \frac{n^4}{z^n} + \cdots + \frac{3^4}{z^3} + \frac{2^4}{z^2} + \frac{1}{z} + \frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \cdots + \frac{z^n}{4^n} + \cdots$$

converges.

Solution. The part of this Laurent series with positive powers of z is a geometric series with ratio $z/4$, so this part of the series converges when $|z/4| < 1$, or $|z| < 4$ (and diverges when $|z| \geq 4$). The part of the series with negative powers of z can be tested by the root test: this part of the series converges when

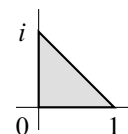
$$\lim_{n \rightarrow \infty} \frac{n^{4/n}}{|z|} < 1$$

(and diverges when this limit is greater than 1). Now $n^{1/n} \rightarrow 1$ when $n \rightarrow \infty$, so $n^{4/n} \rightarrow 1$ too. Therefore this part of the Laurent series converges when $1/|z| < 1$, or $1 < |z|$.

The whole Laurent series converges when both inequalities are satisfied: namely, when $1 < |z| < 4$. The annulus defined by this inequality is the largest annulus in which the whole Laurent series converges.

Extra Credit

When the isosceles right triangle with vertices in the z plane at 0, 1, and i is transformed by the squaring function ($w = z^2$), what is the area of the image region in the w plane? Explain how you know.



Solution. Method 1 The first step is to determine what the image of the triangle looks like.

Squaring is a one-to-one function on the first quadrant that maps points in the first quadrant to points in the upper half-plane (since the angles get doubled). Squaring maps the bottom edge of the triangle back onto the segment $[0, 1]$ of the real axis and maps the left-hand edge of the triangle onto the segment $[-1, 0]$ of the real axis. What about the image of the line segment where $x + y = 1$?

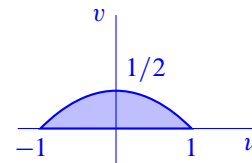
On that line segment, $z = x + iy = x + i(1 - x)$, so

$$z^2 = x^2 - (1 - x)^2 + 2ix(1 - x) = -1 + 2x + 2ix(1 - x).$$

If $z^2 = w = u + iv$, then $u = -1 + 2x$ and $v = 2x(1 - x)$. Eliminate x by using the first equation to say that $x = \frac{1}{2}(1 + u)$. The second equation then implies that

$$v = (1 + u) \left(1 - \frac{1 + u}{2}\right) = (1 + u) \left(\frac{1 - u}{2}\right) = \frac{1 - u^2}{2}.$$

This equation represents a parabola in the u - v plane opening downward. By symmetry, the area under the parabola is twice the area in the first quadrant, which can be computed by integrating the height of the curve (by first-year calculus methods):



$$\text{Area} = 2 \int_0^1 \frac{1 - u^2}{2} du = \left[u - \frac{1}{3}u^3 \right]_0^1 = \frac{2}{3}.$$

Method 2 Since $f(z) - f(z_0) \approx f'(z_0)(z - z_0)$, the local behavior of an analytic function f near a point z_0 is a composition of rotation by angle $\arg f'(z_0)$ and dilation by the factor $|f'(z_0)|$. Since lengths are multiplied locally by $|f'(z_0)|$, areas are multiplied locally by $|f'(z_0)|^2$. In other words, mapping by an analytic function f changes the area element $dx dy$ into $|f'(z)|^2 dx dy$.

Since the derivative of the mapping z^2 is $2z$, the area of the image can be obtained by integrating $|2z|^2$ over the triangle:

$$\text{Area} = \iint_{\text{triangle}} 4(x^2 + y^2) dx dy.$$

The two variables x and y enter symmetrically, so

$$\begin{aligned} \text{Area} &= 8 \iint_{\text{triangle}} x^2 dx dy = 8 \int_0^1 \int_0^{1-y} x^2 dx dy = 8 \int_0^1 \frac{1}{3}(1-y)^3 dy \\ &= \frac{2}{3} [-(1-y)^4]_0^1 = \frac{2}{3}. \end{aligned}$$

The advantage of this second method is that there is no need to determine the image region explicitly.