1. Define each of the following terms.
   (a) proof by mathematical induction
   (b) Cauchy sequence of real numbers
   (c) differentiable function
   (d) divergent infinite series

2. State the following theorems.
   (a) the Bolzano-Weierstrass theorem
   (b) Taylor's theorem
   (c) the Weierstrass M-test

3. Give an example of each of the following.
   (a) a bounded uncountable set
   (b) a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\limsup_{n\to\infty} a_n = 5$ and $\liminf_{n\to\infty} a_n = 3$
   (c) a Riemann integrable function f and a partition of an interval such that the lower sum of f for that partition equals 4
   (d) a power series $\sum_{n=1}^{\infty} a_n x^n$ whose radius of convergence equals 9

4. Prove *one* of the following two theorems.
   (a) A continuous function on a closed, finite interval $[a, b]$ is Riemann integrable.
   (b) The uniform limit of continuous functions is continuous. More precisely, if $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on a set E contained in $\mathbb{R}$, and if $f_n \to f$ uniformly on E as $n \to \infty$, then f is continuous on E.

5. Consider the function f defined on the domain $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} \left(\dfrac{\sin(x)}{x}\right)^2, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

For each of the following properties, state whether or not this function f has the property, and explain why.

(a) continuous on $\mathbb{R}$

(b) uniformly continuous on $\mathbb{R}$

(c) differentiable on $\mathbb{R}$

(d) Riemann integrable on $\mathbb{R}$
(in the sense of improper Riemann integrals)

6. Solve *one* of the following two problems.

(a) Prove that the inequality

$$2\ln(x) < x - \frac{1}{x}$$

holds when $x > 1$. (Here $\ln(x)$ is the natural logarithm function, which may be defined by $\ln(x) = \int_1^x t^{-1}\,dt$.)

(b) Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_1 = 2$, $a_2 = \sqrt{6+2}$, $a_3 = \sqrt{6+\sqrt{6+2}}$, and in general, $a_{n+1} = \sqrt{6+a_n}$ when $n \geq 1$. Prove that the limit $\lim\limits_{n\to\infty} a_n$ exists.