Comments on reading and writing proofs

A student wrote the following.

Theorem 1. *If $x < y$, then $x^2 < y^2$.*

Proof. Multiplying by x, $x < y \Rightarrow x \cdot x < y \cdot x$. Multiplying by y, $x < y \Rightarrow x \cdot y < y \cdot y$. But by commutativity of multiplication, $y \cdot x = x \cdot y$. By transitivity of $<$, $x \cdot x < y \cdot y$. QED □

The student had the right idea, but the presentation has some mistakes. The most serious error is that the theorem is *false*. For example, if $x = -5$ and $y = 1$, then $x < y$, but x^2 is not less than y^2. The theorem would be valid, however, under the additional hypothesis that x is positive.

Some less serious errors are mistakes in style. For example, it is confusing to separate different formulas by just a comma; there ought to be words between formulas to help the reader see where one formula starts and the other ends. Also, symbols like $\Rightarrow$ and $<$ belong in formulas, but not in the middle of sentences as abbreviations for words.

The introductory phrase "Multiplying by x" is a grammatical error known as a "dangling participle". Who or what is doing the multiplying?

Here is a revision of the student's work. Notice that the revised exposition consists mostly of words, with only a few formulas.

Theorem 2. *If x and y are positive real numbers, and $x < y$, then $x^2 < y^2$.*

Proof. It is valid to multiply both sides of an inequality by a positive real number. Apply this principle twice to the hypothesis that $x < y$. Multiply first by x and then by y to deduce that

$$x^2 < yx \qquad \text{and} \qquad xy < y^2.$$

The commutative law for multiplication says that $yx = xy$, so the transitive property of inequality implies the desired conclusion that $x^2 < y^2$. □