Math 409-502

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Announcement

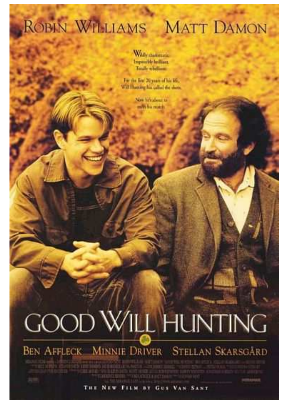
Math Club Movie Social

Good Will Hunting

Blocker 628

Wednesday, December 8, 4:00 PM

FREE SNACKS AND DRINKS



Final examination

The comprehensive final examination will be held in this room on Tuesday, December 14, from 8:00–10:00 AM.

The major topics we covered this semester are:

- limits
- sequences and series
- continuous functions
- differentiable functions
- integrable functions

We covered chapters 1–15, 17–18, the first half of 19–20, and (briefly) 22.1–22.5.

Pointwise versus uniform convergence

Example. For $x > 0$, let $f_n(x) = e^{-nx}$. Then $f_n(x)$ converges to 0 *pointwise* since for each $x > 0$, we have $\lim_{n\to\infty} e^{-nx} = 0$.

The convergence is not uniform because $\sup_{x>0} e^{-nx} = 1$ for every n. If $\epsilon = 1/2$, the definition of uniform convergence cannot be satisfied.

Example: Fourier series

The function $|x|$ cannot be expanded in a Maclaurin series because the function is not differentiable at 0.
The function $|x|$ can, however, be expanded in a series of trigonometric functions: namely (see Example 22.4, p. 315), $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nx)}{n^2}$, when $-\pi \le x \le \pi$.

The series converges uniformly because the tail of the series is less than the tail of the convergent series $\sum 1/n^2$.
This is an example of the *Weierstrass M-test* for uniform convergence of series.

The series represents a continuous function on $(-\infty, \infty)$: the periodic extension of $|x|$ from $[-\pi, \pi]$ to $(-\infty, \infty)$.
The series can be integrated term-by-term.
The series cannot be differentiated term-by-term because the series of derivatives does not converge uniformly.

Example: a nowhere differentiable function

Let $A(x)$ denote the periodic extension of $|x|$ just discussed.

Set $f(x) = \sum_{n=0}^{\infty} \frac{A(2^n x)}{2^n}$.
The series converges uniformly by the Weierstrass M-test, so f is a continuous function.

But there is no point at which f has a derivative.
Roughly speaking, the graph of f has corners in every interval.