Math 409-502

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Announcement

Math Club Meeting

Monday, November 15 at 7:30 PM

Blocker 156

Speaker: Jeff Nash, Aggie former student, from Veritas DGC

Title: Math and Seismic Imaging

FREE FOOD

The derivative and applications

The definition: $f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Some applications:

- the mean-value theorem
- l'Hôpital's rule
- monotonicity; extreme values

The mean-value theorem generalized

Cauchy's form of the mean-value theorem: Suppose f and g are continuous functions on $[a, b]$ that are differentiable on (a, b). Then there exists a point c in (a, b) for which

$$\left(\frac{f(b) - f(a)}{b - a}\right) g'(c) = \left(\frac{g(b) - g(a)}{b - a}\right) f'(c),$$

or $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ if the denominators are non-zero.

Proof. Set $h(x) = \left(\frac{f(b)-f(a)}{b-a}\right) g(x) - \left(\frac{g(b)-g(a)}{b-a}\right) f(x)$.
A computation shows that $h(a) = h(b)$, so by the ordinary mean-value theorem, there is a point c for which $h'(c) = 0$. That reduces to the desired conclusion.

A best-selling author

Guillaume François Antoine Marquis de L'Hôpital
(1661–1704)

Author of the first calculus textbook:
Analyse des infiniment petits

L'Hôpital's rule(s)

If f and g are differentiable functions, and if $\lim_{x\to a} \frac{f(x)}{g(x)}$ is a formally undefined expression of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the second limit exists.
Variations: one-sided limits; $x \to \infty$.

Example. $\lim_{x\to\infty} \frac{\ln(x)}{x}$ is formally $\frac{\infty}{\infty}$. L'Hôpital's rule says that the limit equals $\lim_{x\to\infty} \frac{1/x}{1} = 0$.

Example. $\lim_{x\to 0^+} x\ln(x)$ is formally $0 \cdot (-\infty)$. Rewrite as $\lim_{x\to 0^+} \frac{\ln(x)}{1/x}$. L'Hôpital's rule says that the limit equals $\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$.

Proof of one version of L'Hôpital's rule

Theorem. Suppose f and g are differentiable for large x and $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. If $\lim_{x\to\infty} f'(x)/g'(x) = \infty$, then $\lim_{x\to\infty} f(x)/g(x) = \infty$.

Proof. Fix $M > 0$. We must find b such that $f(x)/g(x) > M$ when $x > b$.

By hypothesis, there is a such that $f(a) > 0$ and $f'(c)/g'(c) > 2M$ when $c \geq a$. Moreover, there is $b > a$ such that $|g(a)/g(x)| < 1/2$ when $x > b$.

Apply Cauchy's mean-value theorem when $x > b$ to get
$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)} > 2M$.
Then $f(x) > f(x) - f(a) > 2M(g(x) - g(a))$, so $f(x)/g(x) > 2M(1 - \frac{g(a)}{g(x)}) > 2M(1 - \frac{1}{2}) = M$.

Homework

1. Read sections 15.2–15.4, pages 212–217.
2. Do Exercise 14.3/2 on page 206.
3. Do Exercise 15.4/2 on page 219.