Math 409-502

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Integration

Some vocabulary

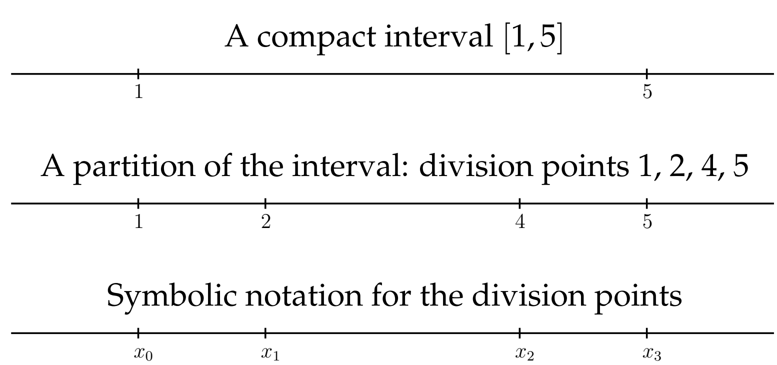
- partition
- mesh
- refinement
- upper sum
- lower sum
- Riemann sum
- integrable function

Some theorems

- Bounded monotonic functions are integrable.
- Bounded continuous functions are integrable.
- Integration and differentiation are inverse operations (fundamental theorem of calculus).

Partition and mesh

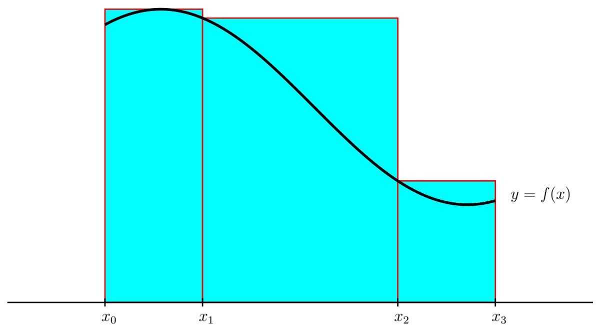
A *partition* of a compact interval $[a, b]$ is a subdivision of the interval.



The *mesh* of a partition is the maximum width of the subintervals. In the above example, the mesh is $4 - 2 = 2$.

Upper sum

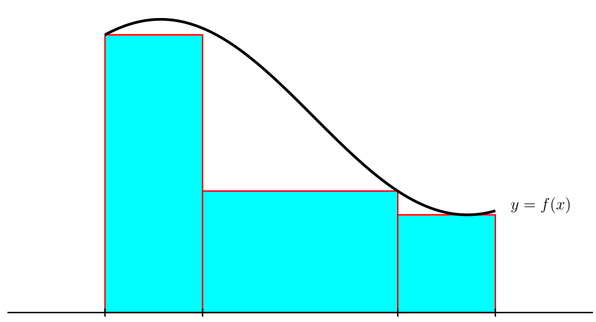
The *upper sum* of a bounded function for a partition of a compact interval means the sum over the subintervals of the supremum of the function on the subinterval times the width of the subinterval.



Symbolic notation: $\sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f(x)$.

Lower sum

The *lower sum* is defined similarly with the infimum in place of the supremum.



Symbolic notation: $\sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f(x)$.

Integrable functions

A function defined on a compact interval $[a, b]$ is *integrable* if (i) the function is bounded, and (ii) for every $\epsilon > 0$, there exists $\delta > 0$ such that for every partition of mesh $< \delta$ the upper sum for the partition and the lower sum for the partition differ by less than ϵ.

Example. A constant function is integrable because every upper sum equals every lower sum.

Example. $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$
is not integrable because every lower sum equals 0, but every upper sum equals the width of the interval.

Homework

- Read sections 18.1 and 18.2, pages 241–244.
- Consider the integrable function $f(x) = x$ on the interval $[1, 2]$. How small must the mesh of a partition be in order to guarantee that the upper sum and the lower sum differ by less than $1/10$?
- Do exercise 18.2/3 on page 248.