Math 409-502

Harold P. Boas
boas@tamu.edu

Definition of the integral

Suppose f is a bounded function on $[a, b]$. The upper sum for any partition is at least as big as the lower sum for any other partition. (Compare both sums to the upper sum and the lower sum for a common refinement.)

Therefore the infimum of the upper sums for all possible partitions is at least as big as the supremum of the lower sums for all possible partitions.

If f is integrable, then these two numbers are equal. The value is defined to be the integral of f, usually denoted by $\int_a^b f(x)\,dx$.

This definition is not usually a convenient way to compute an integral!

Riemann sums

A *Riemann sum* for a bounded function f and for a partition of an interval $[a, b]$ has the form $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$, where each c_k is some point in the subinterval $[x_{k-1}, x_k]$.
Some examples are left-hand endpoint sums, right-hand endpoint sums, and midpoint sums.

For a fixed partition, every Riemann sum is between the upper sum and the lower sum for that partition.

Therefore the integral $\int_a^b f(x)\,dx$ can be computed as a limit of Riemann sums as the mesh size of the partition goes to 0.

Example

Show that $\int_{-1}^{1} \sin(x)\,dx = 0$.
(The same argument will apply to any odd function.)

Since the function $\sin(x)$ is integrable (because $\sin(x)$ is a continuous function), it suffices to consider partitions that are uniformly spaced and symmetric about 0.

No upper sum equals 0 and no lower sum equals 0. But choosing symmetrically located points c_k makes the Riemann sum equal to 0.

Since the integral is a limit of Riemann sums each of which is 0, the integral has the value 0.

Fundamental theorem of calculus

Version 1. If f is integrable on $[a, b]$ and if F is a function whose derivative equals f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof. Consider any partition $a = x_0, x_1, \ldots, x_n = b$ of $[a, b]$. Write $F(b) - F(a)$ as a telescoping sum: $(F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \cdots + (F(x_1) - F(x_0))$.

By the mean-value theorem, the sum equals $F'(c_n)(x_n - x_{n-1}) + F'(c_{n-1})(x_{n-1} - x_{n-2}) + \cdots + F'(c_1)(x_1 - x_0)$, which is the same as $f(c_n)(x_n - x_{n-1}) + f(c_{n-1})(x_{n-1} - x_{n-2}) + \cdots + f(c_1)(x_1 - x_0)$.

This is a Riemann sum for f, so by taking partitions with sufficiently small mesh we get arbitrarily close to $\int_a^b f(x)\,dx$.

Homework

- Read sections 19.1–19.3, pages 251–256.
- Work on proving various versions of l'Hôpital's rule.

Sketch of proof of l'Hôpital's rule

This sketch applies to the case $x \to a$, $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$.
[Remark: f and g need not be defined at a.]

Fix $\epsilon > 0$. By hypothesis, there exists $\delta > 0$ such that $\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon/2$ when $0 < |x - a| < \delta$.

Fix such an x. By hypothesis, there exists x_2 so close to a that $\frac{f(x)}{g(x)}$ differs from $\frac{f(x)-f(x_2)}{g(x)-g(x_2)}$ by less than $\epsilon/2$.

By Cauchy's version of the mean-value theorem, that fraction equals $\frac{f'(c)}{g'(c)}$ for some c. Hence $\frac{f(x)}{g(x)}$ differs from L by less than ϵ when $0 < |x - a| < \delta$.

Technical complication: are all the denominators non-zero?