Math 409-502

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Reminders on radius of convergence

We can find the radius of convergence of a power series by either the ratio test or the root test, but some other test is needed to determine the endpoint behavior.

Useful tests for endpoint behavior are:

- nth-term test
- comparison tests
- alternating series test

Follow-up on endpoint convergence

Last time we saw (by the ratio test) that $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ has radius of convergence equal to 4, and $\sum_{n=1}^{\infty} \frac{n!\, x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ has radius of convergence equal to 2. At the right-hand endpoint, both series become $\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$. That series diverges by the nth-term test. Indeed, $4^n = (1+1)^{2n} = 1 + \binom{2n}{1} + \binom{2n}{2} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{1} + 1$, so $4^n > \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. Thus $\frac{(n!)^2 4^n}{(2n)!} > 1$, so the series cannot converge. For the same reason, divergence occurs at the left-hand endpoint in this example.

Operations on power series

Addition, subtraction, multiplication, and division of power series work the way you expect.

Example

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

so the coefficient of x^5 in the product $\cos(x)\sin(x)$ equals $\frac{1}{5!} + \frac{1}{2!\,3!} + \frac{1}{4!} = \frac{2}{15}$.

Remark on the multiplication theorem

Theorem (page 121): If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge *absolutely*, then the product of the two series equals the absolutely convergent series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Counterexample in case of conditional convergence: Set $a_n = b_n = (-1)^n/\sqrt{n+1}$. Then $\sum a_n$ and $\sum b_n$ are conditionally convergent by the alternating series test, but the series $\sum c_n$ is divergent. Indeed, $c_n = \sum_{k=0}^{n} \frac{(-1)^k(-1)^{n-k}}{\sqrt{k+1}\sqrt{n-k+1}}$. All the terms in this sum have the same sign $(-1)^n$, so $|c_n| \geq \sum_{k=0}^{n} \frac{1}{n+1} = 1$. Hence $\sum c_n$ diverges.

Homework

1. Read Chapter 9, pages 125–134.
2. Do Exercises 9.2/3 and 9.3/1, pages 134–135.