Math 409-502

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Functions

Informal definition

A function is given by a rule or by a formula, for example, $f(x) = x^2$.

Formal definition

A function is a set of ordered pairs with the property that no first element appears more than once, for example, $\{ (a, a^2) : a \in \mathbb{R} \}$.

Aside on groups

In algebra, a *group* is a set equipped with an associative binary operation for which there is an identity element and such that each element has an inverse with respect to the operation.

Examples

The integers form a group under addition.
The number 0 is the identity element, and the additive inverse of n is $-n$.

The non-zero real numbers form a group under multiplication.
The number 1 is the identity element, and the multiplicative inverse of a is $1/a$.

Groups of functions

Do the real-valued functions with domain $\mathbb{R}$ form a group under addition?

Yes: the identity element is the function that is constantly equal to 0, and the inverse of $f(x)$ is $-f(x)$.

Do the non-zero real-valued functions with domain $\mathbb{R}$ form a group under multiplication? It depends on what "non-zero" means.

If "non-zero" means "not equal to the function that is constantly equal to 0", then no: the function $f(x) = x$ does not have a multiplicative inverse that is everywhere defined.

If "non-zero" means "nowhere equal to zero", then yes: the function that is constantly equal to 1 is the multiplicative identity, and the inverse of $f(x)$ is $1/f(x)$.

Composition

Do the real-valued functions with domain $\mathbb{R}$ form a group under composition?

The identity function $f(x) = x$ serves as identity under composition.
But some functions lack inverses.

The function $f(x) = x \sin(x)$ is not one-to-one, so that function does not have an inverse under composition.

The function $g(x) = e^x$ is one-to-one but not onto, so its inverse function $\ln(x)$ is not everywhere defined.

Homework

1. Read sections 10.1 and 10.2, pages 137–142.
2. Do Exercises 10.1/2 and 10.2/1 on page 148.