Math 409-502

Harold P. Boas
boas@tamu.edu

Compactness

Definition

An interval is called *compact* if the interval is both closed and bounded.

Examples

- The interval $[-1, 1]$ is compact.
- The interval $[0, 1)$ is not compact.
  (It is bounded but not closed.)
- The interval $[0, \infty)$ is not compact.
  (It is closed but not bounded.)

The key property is that a sequence of points in a compact interval must have a cluster point that is still in the interval.

Continuous functions on compact intervals

Theorem. *If f is a continuous function on a compact interval, then the range of f is again a compact interval.*

The conclusion of the theorem has three parts.

1. The range is an interval.
2. The range is bounded.
3. The range contains the endpoints of the interval.

- The first part is the Intermediate Value Theorem.
- The second part says that the function has a finite supremum and a finite infimum.
- The third part says that the function attains a maximum value and a minimum value.

Why did the chicken cross the road?

Theorem (Intermediate Value Theorem). *If f is a continuous function on the (compact) interval $[a, b]$, then every number between $f(a)$ and $f(b)$ is in the range of f.*

Example

Show that the function $x^5 - 4x^2 + e^x$ has a zero between $x = 0$ and $x = 1$.

Solution. At $x = 0$ the function has the value 1, and at $x = 1$ the function has the value $-3 + e < 0$. By the theorem, the function takes the intermediate value 0 somewhere in the interval $(0, 1)$.

Using the bisection method, we could locate the zero of the function more precisely.

Proof of the intermediate value theorem

The book gives a proof (page 173) using the Nested Interval Theorem. Here is a different proof.

Without loss of generality, it may be assumed that $f(a) < f(b)$.
We need to show that if k is a number such that $f(a) < k < f(b)$, then the number k is in the range of f.

Let S denote the set of points c with the property that $f(x) < k$ for all x in the interval $[a, c]$. The set S is non-empty because $f(a) < k$ and f is continuous. Let d denote the supremum of S. Claim: $f(d) = k$.

Because f is continuous, $f(d) = \lim_{n\to\infty} f(d - \frac{1}{n})$, so $f(d) \le k$
(by the limit location theorem). It cannot be that $f(d) < k$ (strict inequality), because then there would be points to the right of d in the set S (again by the continuity of f). So the claim holds.

Homework

- Read sections 12.1 and 12.2, pages 172–177.
- Do Exercises 12.1/1 and 12.1/5 on page 180.