Math 409-502

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Summary of convergence tests

- If $a_n \not\to 0$, then $\sum_n a_n$ diverges.
- Comparison tests for positive series: if $a_n \le b_n$ for all large n, or alternatively if a_n/b_n has a finite limit, then convergence of $\sum_n b_n$ implies convergence of $\sum_n a_n$.
- Absolute convergence implies convergence: if $\sum_n |a_n|$ converges, then so does $\sum_n a_n$.
- Ratio and root tests: if either $\lim_{n\to\infty} |a_n|^{1/n}$ or $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists and is strictly less than 1, then $\sum_n a_n$ converges.
- Special tests for *decreasing* positive terms a_n:
  (i) if $f(x) \downarrow 0$ as $x \to \infty$, then $\int^\infty f(x)\,dx$ and $\sum^\infty f(n)$ have the same convergence/divergence behavior (integral test); (ii) if $a_n \downarrow 0$, then $\sum_n (-1)^n a_n$ converges (alternating series test).

Cauchy's condensation test

Another special test for decreasing terms

Suppose $0 < a_{n+1} \le a_n$ for all (large) n. Then the two series $\sum_n a_n$ and $\sum_n 2^n a_{2^n}$ either both converge or both diverge.

Example: $\sum_n \frac{1}{n\ln(n)}$

Since $\frac{1}{n\ln(n)}$ is a decreasing function of n, the test applies and says that the convergence/divergence behavior is the same as for the series $\sum_n 2^n \frac{1}{2^n \ln(2^n)}$. That simplifies to $\sum_n \frac{1}{n\ln(2)}$, which is a multiple of the divergent harmonic series. Therefore the original series $\sum_n \frac{1}{n\ln(n)}$ diverges too.

Proof of the condensation test (sketch)

$$\begin{aligned} a_8 + a_9 + \cdots + a_{15} &\le 8a_8 \le 2(a_4 + a_5 + a_6 + a_7) \\ a_{16} + a_{17} + \cdots + a_{31} &\le 16a_{16} \le 2(a_8 + a_9 + \cdots + a_{15}) \\ &\vdots \end{aligned}$$

Adding such inequalities shows that partial sums of $\sum_n 2^n a_{2^n}$ are bounded below by partial sums of $\sum_n a_n$ and are bounded above by twice the partial sums of $\sum_n a_n$. Therefore the two series have the same convergence/divergence behavior.

Power series

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^n\sqrt{n}}$

For which values of x does that series converge?
[This is Exercise 8.1/1a on page 123.]

Solution:
By the root test, the series converges (absolutely) when

$$1 > \lim_{n\to\infty} \left| \frac{x^n}{2^n\sqrt{n}} \right|^{1/n} = \lim_{n\to\infty} \frac{|x|}{2\sqrt{n^{1/n}}} = \frac{|x|}{2},$$

that is, when $|x| < 2$.
The series diverges when $|x| > 2$ by the (proof of the) root test.
A different test is needed to see what happens when $x = \pm 2$.

Homework

- Read section 8.1, pages 114–117.
- Do Exercise 7.6/1a,c on page 111.
- Do Exercise 8.1/1g on page 123.