Math 409-502

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Limit theorems

Sums, products, and quotients

If $a_n \to L$ and $b_n \to M$ then $a_n + b_n \to L + M$;
and $a_n \cdot b_n \to L \cdot M$;
and if in addition $L \neq 0$ then $b_n / a_n \to M/L$.
[In the third case, $a_n \neq 0$ when n is large, so b_n / a_n makes sense for n large.]

Squeeze theorem

If $a_n \leq b_n \leq c_n$ for all sufficiently large n, and if the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ both converge to the same limit, then the sequence $\{b_n\}_{n=1}^{\infty}$ converges, and to the same limit.

Limit theorems continued

Location theorems

If $a_n \to L$ and if $a_n < M$ for all sufficiently large n, then $L \leq M$.
The example $a_n = n/(n+1)$ shows that we cannot draw the conclusion $L < M$.

If $a_n \to L$ and $L < M$, then $a_n < M$ for all sufficiently large n.
The example $a_n = (n+1)/n$ shows that the hypothesis $L \leq M$ is insufficient.

Subsequences

A non-convergent sequence may have convergent *subsequences*.

Example:

$$\frac{1}{4}, \frac{3}{4}, 2, \frac{1}{8}, \frac{7}{8}, 4, \frac{1}{16}, \frac{15}{16}, 8, \frac{1}{32}, \frac{31}{32}, 16, \ldots$$

If a sequence converges, however, then every subsequence converges to the same limit.

Example: $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \ldots$ is a subsequence of the convergent sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$, so it converges to 0.

Homework

1. Read sections 5.4 and 5.5, pages 68–73.
2. Do Problem 5-7, page 75.