Math 409 Exam 1 Spring 2010

Advanced Calculus I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "determine" or "true/false" or "give an example" requires a supporting explanation. Please explain your reasoning in complete sentences.

1. Determine the largest natural number n such that $|n^2 - 2| < 18$.

2. Let E denote the set of real numbers having decimal expansions

$$0.a\,b\,c\,d\,\ldots$$

that do *not* contain the digit 5. Determine the supremum of the set E.

3. Suppose that a sequence $\{x_n\}_{n\in\mathbf{N}}$ of real numbers is defined recursively as follows:

$$\begin{aligned} x_1 &= 1, \qquad \text{and} \\ x_{n+1} &= \sqrt{6 + x_n} \qquad \text{when } n \in \mathbf{N}. \end{aligned} \tag{†}$$

(The square-root symbol indicates the *positive* square root.)

Prove that $x_n < 3$ for every natural number n.

4. Give an example of a bijective function $f\colon \mathbf{R} \to (0, \infty)$.
[Recall that "bijective" means both one-to-one and onto.]

5. True or false: If E is an uncountable subset of $\mathbf{R}$, then the complement $\mathbf{R} \setminus E$ is a countable set.

6. (a) State the definition of what "$\lim_{n\to\infty} x_n = L$" means.

   (b) Use the definition to prove that $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$.

7. Prove that the sequence (†) defined in problem 3 converges.
[For the purposes of this problem, you may assume that the conclusion of problem 3 is valid. The value of $\lim_{n\to\infty} x_n$ actually is equal to 3, but you are not required to prove that.]