

Advanced Calculus I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says “determine” or “true/false” or “give an example” requires a supporting explanation. Please explain your reasoning in complete sentences.

1. If x_1, x_2, \dots is a Cauchy sequence of real numbers, is it necessarily true that $|x_1|, |x_2|, \dots$ is a Cauchy sequence too? Give a proof or a counterexample, whichever is appropriate.
2. (a) State the definition of what “ $\lim_{x \rightarrow 0} f(x) = 0$ ” means.
 (b) Use the definition to prove that $\lim_{x \rightarrow 0} e^{-1/x^2} = 0$.
3. Evidently $2^x = x^2$ when $x = 2$ and when $x = 4$. Are there any negative values of the real number x for which $2^x = x^2$? Explain how you know. [You may assume that 2^x is an everywhere differentiable function of x .]
4. If $f(x) = \sin(x)$ for every real number x , is the function $f: \mathbf{R} \rightarrow \mathbf{R}$ uniformly continuous on \mathbf{R} ? Explain why or why not.
5. Suppose that

$$f(x) = \begin{cases} x \cos(1/x), & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

Is the function f differentiable at the point where $x = 0$? Explain why or why not.

6. Suppose $f: \mathbf{R} \rightarrow \mathbf{R}$ is a differentiable function, and $\lim_{x \rightarrow \infty} f'(x) = 3$. Determine $\lim_{x \rightarrow \infty} (f(x+2) - f(x))$.
7. Suppose $f(x) = \frac{2}{1+x}$ for every positive real number x , and let g denote the iterated composition $\underbrace{f \circ f \circ \dots \circ f}_{409 \text{ copies of } f}$. Determine the derivative $g'(1)$.